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Spectral Properties and Boundary Conditions for Second Order Elliptic Operators

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Abstract

This article investigates the regular solvability of boundary value problems for second-order elliptic operator-differential equations with discontinuous coefficients and non-standard boundary conditions in a Hilbert space setting. The authors focus on operator equations defined over a finite interval and aim to identify the structural and spectral conditions under which these problems admit unique, stable solutions. They construct the problem using self-adjoint and positively defined operators, incorporating piecewise-constant coefficients and boundary operators. The study introduces a specialized function space to accommodate the differential and boundary conditions and establishes conditions ensuring the existence, uniqueness, and continuity of solutions with respect to input data. By analyzing the properties of associated linear operators and leveraging classical results from functional analysis, such as the Banach inverse operator theorem, the authors demonstrate that the solution operator is bounded and invertible under specific spectral constraints. This work contributes to the theoretical understanding of elliptic operator-differential equations and provides valuable tools for further analysis in mathematical physics, engineering, and applied mathematics contexts where such boundary value problems frequently arise.

Keywords: Operator-differential equation, Boundary value problem, Hilbert space, Self-adjoint operator, A regular solution.

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1|Introduction

Boundary value problems for operator-differential equations have emerged as one of the central topics in the study of functional analysis and partial differential equations due to their wide applicability in theoretical and applied sciences. These problems, particularly in the context of second-order elliptic operators, arise naturally in physics, engineering, and mathematical modeling of various diffusion, elasticity, and wave propagation processes. The mathematical challenges associated with such problems become even more pronounced when the equations involve discontinuous coefficients, complex boundary conditions, and are posed in infinite-dimensional Hilbert spaces. The development of the theory for boundary value problems involving elliptic operator-differential equations has a rich history. Fundamental groundwork was laid by Krein [12], Dubinsky [7], and Yakubov [21], who investigated differential equations in Banach and Hilbert spaces. These studies introduced the analytical machinery necessary to work with abstract differential operators and their spectral properties. Subsequent contributions by Lions and Magenes [13] further established the role of boundary conditions in determining the solvability and regularity of such problems. Their work emphasized the importance of compatibility conditions between operator domains and boundary spaces, which laid the foundation for later generalizations. In particular, the solvability of elliptic equations with operator coefficients has been thoroughly explored by researchers such as Agayeva [1]-[3], Mirzoyev [14, 15, 19], and Aliyev [4, 5]. These studies considered boundary value problems where the boundary conditions themselves include operator terms, making the problem more intricate. They introduced the concept of regular solvability, which guarantees that for every suitable input (forcing term), a unique solution exists that continuously depends on the input. This notion is crucial in ensuring the stability of solutions and their practical applicability. The use of Hilbert space settings in such investigations allows for a more general and flexible framework. The inner product structure of Hilbert spaces enables the deployment of powerful tools such as spectral theory, compactness arguments, and energy estimates. Within this framework, the study of operator-differential equations with discontinuous or piecewise constant coefficients becomes tractable, especially when combined with appropriate function spaces and domain characterizations, as explored by Gasimov [8], Gasimova [9, 10], and others. Despite the progress, there remain significant challenges in fully characterizing the solvability of such equations, particularly when the coefficients exhibit discontinuities or when boundary operators act between different scales of Hilbert spaces. The introduction of discontinuous coefficients, for example, reflects physical scenarios such as media with layered or composite structures, where the governing equations change behavior at interface points. Such complexity necessitates refined analytical techniques and carefully constructed function spaces, such as weighted Sobolev spaces or abstract interpolation spaces, which accommodate the irregular behavior of solutions.

The present study, aims to contribute to this ongoing development by examining a class of second-order elliptic operator-differential equations defined on a finite interval, where the principal part of the operator is accompanied by lower-order terms, and the boundary conditions are non-classical and include operator expressions. The key novelty of the paper lies in identifying sufficient spectral and structural conditions under which the boundary value problem is regularly solvable. In other words, it ensures that the associated solution operator is bounded and invertible, and that the solutions depend continuously on the given data in the appropriate norm. Building on earlier work by Aghayeva [1]-[3] and her collaborators [14, 15], this paper extends the theory to settings where the boundary operators may not be self-adjoint or compact, and the coefficients may exhibit jumps or discontinuities. It also incorporates recent advances in the use of interpolation theory and operator semigroups, as discussed by researchers such as Gorbacuk and Gorbacuk [11], and Mirzoyev et al. [16]-[18]. The authors provide rigorous proofs demonstrating that under appropriate inequalities involving the norms of the boundary operators and spectral gaps of the principal operator, the problem admits a unique, stable solution. In addition, the authors make use of functional analytic techniques such as the Banach inverse mapping theorem and energy-type estimates to establish the main results. This approach not only guarantees existence and uniqueness but also yields important a priori estimates for the solution. These estimates are crucial for numerical methods and applications, as they provide bounds on the solution in terms of the input data. In conclusion, this study significantly enriches the current understanding of elliptic operator-differential equations with discontinuous coefficients and operator boundary conditions. By establishing new solvability criteria and offering detailed proofs within the Hilbert space framework, the paper contributes both theoretical insights and practical tools for further exploration in the analysis of partial differential equations and their applications in mathematical physics. This study builds upon recent advancements in numerical and analytical methods for solving differential and integral equations. In particular, the comparison of numerical methods for solving ODEs, Volterra integral, and integro-differential equations by Aghayeva et al. [22], and the comparative study of Adam's methods with other multistep approaches in initial-value problems by Shafiyeva et al. [23], provide a methodological backdrop for addressing the regular solvability of operator-differential equations. Moreover, the utilization of pseudospectral techniques for fractional equations, as demonstrated in the work by Liu et al. [24], and the exploration of the Cauchy problem for higher-order elliptic equations by Niyozov et al. [25], underscore the ongoing relevance of spectral properties and boundary behavior in complex differential models. These references collectively support the framework and innovations presented in this paper.

Let H be a separable Hilbert space, A a positively defined self-adjoint operator in H with the domain of definition $D(A)$. Denote by H_α the scale of Hilbert spaces generated by the operator A, i.e. $H_\alpha = D(A^\alpha)\,,\ (x,y)_\alpha = (A^\alpha x, A^\alpha y)\,,\quad x,y \in H_\alpha\,,\ \alpha \geq 0$. For $\alpha = 0$ we assume $H_0 = H$.

Let $L_2((0,T);H)$ be the Hilbert space of all functions$f(t)$, defined almost everywhere on the interval $(0,T)$ with values in H, such that

$$\| f \|_{L_2((0,T);H)} = \left(\int_0^T \|f(t)\|^2 \, dt \right)^{\frac{1}{2}} < \infty.$$

Following the monographs [1, 2], we define the Hilbert space $W_2^2((0,T);H) = \{u : u'' \in L_2((0,T);H),\ A^2 u \in L_2((0,T);H)\}$

with the norm

$$\| u \|_{W_2^2((0,T);H)} = \left(\|u''\|^2_{L_2((0,T);H)} + \left\|A^2 u\right\|^2_{L_2((0,T);H)} \right)^{\frac{1}{2}}.$$

Similarly, the spaces $L_2(R;H)$ and $W_2^2(R;H)$, where R is the real line.

We introduce the following subspace of $W_2^2((0,T);H)$

$$W_{2,K}^2((0,T);H) = \{u : u \in W_2^2((0,T);H),\ u'(0) = Ku(0)\,,\quad u'(T) = 0\},$$

where $K \in L(H_{3/2}, H_{1/2})$. Here, $L(X,Y)$ is the space of bounded linear operators acting from X into Y. In what follows, derivatives are understood in the sense of distributions

Consider the boundary value problem in the space H

$$-u''(t) + \rho(t)A^2 u(t) + A_1 u'(t) + A_2 u(t) = f(t)\,,\quad t \in (0,T), \tag{1}$$

$$u'(0) = Ku(0), \quad u'(T) = 0, \tag{2}$$

where $f(t), u(t)$ are functions with values in$H$, and the operator coefficients satisfy the following conditions:

1. A is a positively defined self-adjoint operator in H;

$$2.\rho(t) = \begin{cases} \alpha^2\,, & t \in (0,t_0)\,, \\ \beta^2\,, & t \in (t_0,T)\,,\ t_0 \in (0,T)\,,\ \alpha > 0\,,\ \beta > 0\,; \end{cases}$$

$$3.K \in L(H_{3/2}, H_{1/2});$$

4. $B_j = A_j A^{-j}$ is bounded in $H\,,\ j = 1,2$.

Definition 1. If for $f(t) \in L_2((0,T);H)$ there exists a vector-function $u(t) \in W_2^2((0,T);H)$, that satisfies equation (1) almost everywhere on $(0,T)$, the $u(t)$ is called a regular solution of equation (1).

Definition 2. If for any $f(t) \in L_2((0,T);H)$ there exists a regular solution $u(t)$ of equation (1) satisfying the boundary conditions (2) in the sense of convergence

$$\lim_{t\to+0} \|u'(t) - Ku(t)\|_{1/2} = 0\,, \lim_{t\to T-0} \|u'(t)\|_{1/2} = 0$$

and the inequality $\| u \|_{W_2^2((0,T);H)} \leq const \| f \|_{L_2((0,T);H)}$ holds, then the problem (1), (2) is called regularly solvable.

In this work, we provide **sufficient conditions** on the coefficients of the equation and the boundary conditions which ensure **regular solvability** of problem (1), (2). It should be noted that boundary value problems for elliptic operator-differential equations of second order are studied, for example, in works [1].

Let us denote:

$$P_0 u = -u'' + \rho(t) A^2 u\,, \;\; P_1 u = \sum_{j=0}^{1} A_{2-j} u^{(j)}, \;\; u \in W^2_{2,K}((0,T);H)$$

and

$$Pu = P_0 u + P_1 u, \;\; u \in W^2_{2,K}((0,T);H).$$

First, we investigate the solvability of Equation $P_0 u = f$.

2|Some Results

Lemma 1. Let conditions 1)–3) hold and $ReA^{-1}K \geq 0$ and $H_{3/2}$. Then for any $u \in W^2_{2,K}((0,T);H)$ the following inequality holds:

$$\|Au'\|^2_{L_2((0,T);H)} + \left\|\rho^{1/2} A^2 u\right\|^2_{L_2((0,T);H)} \leq Re(P_0 u, A^2 u)_{L_2((0,T);H)}. \tag{3}$$

Proof. After multiplying the equation $P_0 u = f$ scalarly by the function $A^2 u$ in space $L_2((0,T);H)$ we have

$$-Re(u'', A^2 u)_{L_2((0,T);H)} + \left\|\rho^{1/2} A^2 u\right\|^2_{L_2((0,T):H)} = Re(P_0 u, A^2 u)_{L_2((0,T);H)}.$$

After integration by parts, we obtain

$$Re(A^{1/2}u'(0), A^{3/2}u(0)) + \|Au'\|^2_{L_2((0,T);H)} + \left\|\rho^{1/2} A^2 u\right\|^2_{L_2((0,T);H)} =$$

$$= Re(P_0 u, A^2 u)_{L_2((0,T);H)},$$

or

$$Re(A^{-1}Ku(0), u(0))_{3/2} + \|Au'\|^2_{L_2((0,T);H)} + \left\|\rho^{1/2} A^2 u\right\|^2_{L_2((0,T);H)} =$$

$$= Re(P_0 u, A^2 u)_{L_2((0,T);H)}. \tag{4}$$

Considering that $ReA^{-1}K \geq 0$ in $H_{3/2}$, of equality (4) we obtain the statement of the lemma.

Corollary 1. Under the conditions of the lemma, the homogeneous equation $P_0 u = 0$ only the trivial solution.

Corollary 2. Under the lemma's conditions, for any $u \in W^2_{2,K}((0,T);H)$ the following inequalities hold:

$$\left\|A^2 u\right\|_{L_2((0,T);H)} \leq \frac{1}{\min(\alpha^2;\beta^2)} \left\|P_0 u\right\|_{L_2((0,T);H)},$$

$$\left\|Au'\right\|_{L_2((0,T);H)} \leq \frac{1}{2\min(\alpha;\beta)} \left\|P_0 u\right\|_{L_2((0,T);H)}.$$

Proof. It follows from inequality (3) that at $u(t) \in W^2_{2,K}((0,T);H)$ inequality is true $\left\|\rho^{1/2} A^2 u\right\|^2_{L_2((0,T):H)} \leq \left\|P_0 u\right\|_{L_2((0,T);H)} \cdot \left\|A^2 u\right\|_{L_2((0,T);H)}$. Hence we have:

$$\left\|A^2 u\right\|^2_{L_2((0,T);H)} \leq \max_t \rho^{-1}(t) \left\|\rho^{1/2} A^2 u\right\|^2_{L_2((0,T);H)} \leq$$

$$\leq \frac{1}{\min(\alpha^2,\beta^2)} \left\|P_0 u\right\|_{L_2((0,T);H)} \cdot \left\|A^2 u\right\|_{L_2((0,T);H)}.$$

Here, $\left\|A^2u\right\|_{L_2((0,T);H)} \leq \frac{1}{\min(\alpha^2,\beta^2)} \left\|P_0u\right\|_{L_2((0,T);H)}$. On the other hand, for any $\varepsilon > 0$ it follows from inequality (3) that

$$\left\|Au'\right\|^2_{L_2((0,T);H)} + \left\|\rho^{1/2}A^2u\right\|^2_{L_2((0,T);H)} \leq \left\|\rho^{1/2}A^2u\right\|_{L_2((0,T);H)} \left\|\rho^{-1/2}P_0u\right\|_{L_2((0,T);H)} \leq$$

$$\leq \frac{\varepsilon}{2}\left\|\rho^{1/2}A^2u\right\|^2_{L_2((0,T);H)} + \frac{1}{2\varepsilon}\left\|\rho^{-1/2}P_0u\right\|^2_{L_2((0,T);H)}.$$

Assume that $\varepsilon = 2$, we get

$$\left\|Au'\right\|^2_{L_2((0,T);H)} \leq \frac{1}{4}\left\|\rho^{-1/2}P_0u\right\|^2_{L_2((0,T);H)},$$

or

$$\left\|Au'\right\|_{L_2((0,T);H)} \leq \frac{1}{2\min(\alpha,\beta)}\left\|P_0u\right\|_{L_2((0,T);H)}.$$

Investigation proven.

3|Main Results

Theorem 1. Let the conditions of lemma 1)–3) be satisfied. Then the operator P_0 is isomorphic to space$W^2_{2,K}((0,T);H)$ on $L_2((0,T);H)$.

Proof. It follows from Corollary 1 that the equation $KerP_0 = \{0\}$. Let's prove that the equation $P_0u = f$ Let us prove that the equation has a solution at any $f(t) \in L_2((0,T);H)$. It is easy to see that the function

$$\alpha_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(\xi^2E + \alpha^2A^2)^{-1}\int_0^T f(s)e^{i\xi(t-s)}dsd\xi$$

and

$$\beta_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(\xi^2E + \beta^2A^2)^{-1}\int_0^T f(s)e^{i\xi(t-s)}dsd\xi$$

belong to the space $W^2_2(R;H)$ and satisfy the equations $u'' + \alpha^2A^2u = f$ and $-u'' + \beta^2A^2u = f$ in $(0,T)$ almost everywhere, respectively. Then it is obvious that $\alpha_1(t), \beta_1(t) \in W^2_2((0,T);H)$ and by the trace theorem $\alpha_1^{(i)}(0)$, $\beta_1^{(i)}(0)$, $\alpha_1^{(i)}(T)$,$\beta_1^{(i)}(T) \in H_{2-j-1/2}$ $(j = 0,1)$ [2]. We will search for solutions of equation $P_0u = f$ in form

$$u(t) = \begin{cases} \alpha_1(t) + e^{-\alpha tA}\varphi_1 + e^{\alpha(t-t_0)A}\varphi_2, & t \in (0,t_0), \\ \beta_1(t) + e^{\beta(t-T)}\varphi_3 + e^{\beta(t_0-t)A}\varphi_4, & t \in (t_0,T), \end{cases}$$

where vectors $\varphi_j \in H_{3/2}$, $j = \overline{1,4}$ belong to the definition. From condition (2) and from the equations $u(t_0-0) = u(t_0+0)$,$u'(t_0-0) = u'(t_0+0)$ $(u \in W^2_{2,K}((0,T);H)$ relatively φ_j, $j = \overline{1,4}$, we obtain system of equations. Considering Corollary 1, from these conditions the vectors φ_j, $j = \overline{1,4}$ are defined. Thus, $u \in W^2_{2,K}((0,T);H)$ and $P_0u = f$. In other hand, $\left\|P_0u\right\|_{L_2((0,T);H)} \leq \sqrt{2}\left\|u\right\|_{W^2_2((0,T);H)}$. Then the statement of the theorem follows from Banach's theorem on the inverse operator. The theorem is proved.

Now let us prove the main result of the paper.

Theorem 2. Let conditions 1)–4) be satisfied, $\mathrm{Re}A^{-1}K \geq 0$ in $H_{1/2}$ and there is an inequality

$$q(\alpha,\beta) = \frac{1}{2\min(\alpha,\beta)}\left\|B_1\right\| + \frac{1}{\min(\alpha^2,\beta^2)}\left\|B_2\right\| < 1.$$

Then the problem (1), (2) is regularly solvable.

Proof. Let us write the problem (1), (2) in the form of the equation $Pu = P_0u + P_1u$, $u \in W^2_{2,K}((0,T);H)$, $f \in L_2((0,T);H)$. After replacement $P_0u = \omega$ we get equation $\omega + P_1P_0^{-1}\omega = f$ in space $L_2((0,T);H)$. Applying Corollary 2, we obtain that for any $\omega \in L_2((0,T);H)$.

$$\left\|P_1P_0^{-1}\omega\right\|_{L_2((0,T);H)} = \left\|P_1u\right\|_{L_2((0,T);H)} \leq \left\|B_1\right\| \ \left\|Au'\right\|_{L_2((0,T);H)} +$$

$$+\left\|B_2\right\| \ \left\|A^2u\right\|_{L_2((0,T);H)} \leq q(\alpha,\beta)\left\|P_0u\right\|_{L_2((0,T);H)} = q(\alpha,\beta)\left\|\omega\right\|_{L_2((0,T):H)}.$$

Since $q(\alpha,\beta)<1$, then $E+P_1P_0^{-1}$ convert in the space and $u=P_0^{-1}(E+P_1P_0^{-1})^{-1}f$. Hence, it follows that $\| u \|_{W_2^2((0,T);H)} \leq const \| f \|_{L_2((0,T);H)}$. The theorem is proved.

Corollary 3. Let conditions 1) and 4) be satisfied and inequality holds $\frac{1}{2}\|B_1\|+\|B_2\|<1$. Then equestion

$$-u''(t)+A^2u(t)+A_1u'(t)+A_2u(t)=f(t), \quad t\in(0,T),$$

$$u'(0)=0, \quad u'(T)=0$$

regularly solvable.

The proof follows from Theorem 2 when $\rho(t)\equiv 1$ and $K=0$. This issue was examined in work [5].

4|Conclusion

In this study, we have addressed the regular solvability of boundary value problems for a class of second-order elliptic operator-differential equations with discontinuous coefficients and operator-type boundary conditions in a Hilbert space framework. These types of problems arise in many theoretical and applied contexts, especially where the physical systems modeled involve media with piecewise properties or require the inclusion of abstract boundary interactions. The central contribution of the paper is the establishment of sufficient conditions under which such boundary value problems are not only solvable but regularly solvable. That is, for each admissible input function, a unique solution exists within a specifically defined function space, and this solution depends continuously on the input data. This guarantees the stability and robustness of solutions, which are essential for both theoretical analysis and computational implementation. Our results generalize and extend previous work in the field by accommodating discontinuous operator coefficients and more general boundary operators that act between different levels of Hilbert space scales. The proofs rely on the application of advanced tools from functional analysis, including spectral theory, the theory of bounded linear operators, and the Banach inverse operator theorem. In particular, we derived and utilized operator inequalities that ensure the invertibility of the solution operator and provided energy-type estimates that quantify the dependence of the solution norm on the norm of the input function. Furthermore, by reformulating the original boundary value problem into a system involving auxiliary operators and establishing the isomorphism of the principal part, we created a foundation for future research. Our approach can be adapted to more complex systems, including those with time dependence, nonlinear perturbations, or variable domain geometries.

In conclusion, the findings of this article contribute significantly to the broader theory of operator-differential equations, providing new pathways for analyzing boundary value problems with complex structural and spectral characteristics. This work not only reinforces the applicability of Hilbert space methods in differential equations but also lays the groundwork for future studies aimed at exploring numerical methods, control problems, and real-world applications in engineering and physics.

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Author Contribution

G.A. Aghayeva: methodology, software, and editing. D.A. Juraev: conceptualization, writing and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare that there is no conflict of interest in relation to the reported research findings. The sponsors did not play a role in the design of the study, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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