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On a Derivation of the Classical Differential Equation for Chebyshev Polynomials of the First Kind

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Abstract

In this study, the derivation of the second-order differential equation for first-order Chebyshev polynomials is revisited using orthogonality theory and the Pearson equation approach. By employing the Pearson-type relation associated with the Chebyshev weight function, the relationship between the weight function and the corresponding differential operator is systematically analyzed. This approach leads directly to the classical second-order differential equation satisfied by Chebyshev polynomials and reveals the Sturm–Liouville structure of the associated operator. The interplay between orthogonal polynomials systems and second-order differential equations is emphasized, highlighting the analytical structure underlying these polynomials. In particular, the Pearson equation framework provides an alternative and systematic derivation of the differential equation governing the Chebyshev polynomial system

Keywords and phrases: Chebyshev polynomials, Orthogonal polynomials, Pearson equation, Second-order differen tial equations, Sturm–Liouville theory, Spectral methods, Orthogonality, Approximation theory.

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1| Introduction

The theory of orthogonal polynomials has remained one of the most active areas of research in mathematical analysis for nearly two centuries. Its foundations were laid by mathematicians such as Legendre, Chebyshev, Hermite, and Laguerre, and it was subsequently enriched through its deep connections with the theory of differential equations developed by Sturm and Liouville. In partic ular, the fact that many classical orthogonal polynomials arise as solutions of certain second-order linear differential equations demonstrates that these polynomials are not merely algebraic objects but also possess rich analytical structures.
Consequently, the theory of orthogonal polynomials has become an indispensable field of research in both pure and applied mathematics. Comprehensive treatments of the theory can be found in the classical monographs of Chihara [1], Szegő [6], and Suetin [5]. The fundamental relationship between orthogonal polynomial systems and second-order differential equations has played a significant role in the development of modern approximation theory and mathematical physics. Within the classical family of orthogonal polynomials, Chebyshev polynomials hold a special place. Introduced by Pafnuty Lvovich Chebyshev in the 19th century during the study of best approximation problems, these polynomials play a fundamental role in approximation theory, par ticularly due to their minimax properties. Chebyshev polynomials are widely used today in many fields such as interpolation theory, numerical integration, spectral methods, signal processing, con trol theory, and engineering applications. Their advantages in reducing the Runge effect, which occurs in high-order polynomial interpolation, and in determining optimal nodal points, make these polynomials extremely valuable from a numerical analysis perspective. The importance of orthog onality in function spaces has also motivated investigations in generalized Morrey-type spaces and related settings. In particular, several recent studies have considered the construction of orthogonal polynomial systems and operator-theoretic properties in generalized function spaces [7, 8, 9, 10].
These developments indicate that classical orthogonal polynomials continue to play an important role not only in approximation theory but also in contemporary functional analysis. One of the remarkable properties of Chebyshev polynomials is that they are solutions to a given second-order differential equation. This differential equation forms a direct bridge between the orthogonality properties of polynomials and Sturm–Liouville theory. Indeed, the vast major ity of classical orthogonal polynomials are obtained as eigenfunctions of Sturm–Liouville problems defined under appropriate weighting functions. Therefore, the study of the differential equation provided by a family of orthogonal polynomials is not only a matter of theoretical interest but also of great importance in understanding the spectral, analytic, and approximate computational properties of these polynomials. A general theory of classical orthogonal polynomials and their associated differential equations was later developed by Nikiforov, Suslov, and Uvarov [11], while further extensions involving special functions and hypergeometric structures were investigated by Andrews, Askey, and Roy [12] as well as Koekoek, Lesky, and Swarttouw [15]. More recently, Is mail [13] provided a comprehensive modern treatment of orthogonal polynomial systems and their analytical foundations. Detailed accounts of their analytical and computational properties can be found in the books of Mason and Handscomb [17] and Boyd [18]. Furthermore, the role of Chebyshev polynomials in spectral approximations and numerical solutions of differential equations has been thoroughly investigated in the literature [16]. The study of differential equations satisfied by orthogonal polynomial systems is also closely related to the broader theory of ordinary differential equations and Sturm–Liouville operators. Comprehensive discussions of such differential operators and their spectral properties may be found in the monograph of Teschl [19]. Moreover, many of the special functions and orthogonal polynomial families arising as solutions of second-order differential equations are systematically documented in the NIST Handbook of Mathematical Functions [20],which serves as one of the standard references in the field.
Pearson equations are used as an important tool in explaining the relationship between or thogonal polynomials and differential operators. Pearson-type weight functions establish a natural connection between polynomial systems satisfying orthogonality conditions and the differential operators associated with these systems. This approach allows for the systematic derivation of the differential properties of classical orthogonal polynomials and provides a better understanding of the analytic structure of weight functions. The connection between orthogonal polynomial systems and differential operators is closely related to the theory of Pearson equations and Sturm–Liouville problems. Early contributions in this direc tion were made by Geronimus [2, 3], who investigated the characterization of orthogonal polynomial sequences through moment problems and associated weight functions. Similar developments may also be found in the work of Korous [4]. These studies established a framework in which orthogonality properties can be linked to second-order differential equations through suitable weight functions. The main motivation of this study is to re-examine the second-order differential equation

satisfied by Chebyshev polynomials from the perspective of orthogonality theory and Pearson equations, and to reveal the structural relationship between these two approaches. In this study, the relevant differential operator is obtained by utilizing the Pearson-type characterization of the Chebyshev weight function, and the classical differential equation satisfied by Chebyshev polynomials is derived using this operator. Thus, the connection between orthogonal polynomial theory, Pearson equations, and Sturm–Liouville structures is addressed from a holistic perspective. In this respect, the study goes beyond merely reproducing a known result; it contributes to explaining the analytic structure of Chebyshev polynomials through a different method and helps to understand the relationships between fundamental concepts of orthogonal polynomial theory more clearly.

2| Basic Information and Statement of the Orthogonal Polynomials

In this section we recall the definition and the basic properties of orthogonal polynomials that can be found in the basic literature on orthogonal polynomials (cf. [4], [5], and [6]).These classical orthogonal polynomials satisfy an orthogonality relation, a three term recurrence relation, a second order linear differential equation and a so-called Rodrigues formula. Moreover, for each family of classical orthogonal polynomials we have a generating function.

De˝nition 1. Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of polynomials, where every polynomial $P_n(x)$ has the degree n. If for all polynomials of this system $\int_a^b h(x)\,P_n(x)P_m(x)dx = 0, n \neq m$ then the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ called orthogonal in (a, b) with respect to the weight function $h(x)$. If moreover

$$\|P_n(x)\|_{h(x)} = \left(\int_a^b h(x)P_n^2(x)dx\right)^{2} = 1$$

for every $n = 0, 1, 2, ...$, then the polynomials are called orthonormal in (a, b). So the condition of the orthonormality of the system $\{P_m(x)\}_{n=0}^{\infty}$ has the form $\int_a^b h(x)\,P_n(x)\,P_m(x)\,dx = \delta_{mn}$, where δ_{mn} is Kronecker delta which is defined by $\delta_{mn} := \begin{cases} 0, m \neq n \\ 1, m = n \end{cases}$ for $m, n = \{0, 1, 2 \ldots\}$. Jacobi polynomials can be defined by means of their Rodrigues formula and it is stated below

$$P_n^{(;\ )}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-}\ (1+x)^{-}\ D^n\left[(1-x)^{n+}\ (1+x)^{n+}\right] \quad for\ n = 0, 1, 2, \ldots. \ (2)$$

In the sequel we will often use and the notation $D = \frac{d}{dx}$ for diferentiation operator. Then we have Leibniz' rule. Whenever you typeset mathematical notation, it needs to be inside a mathematics environment. Displayed mathematical equations is aligned to the center and the equation numbering is given in order, e.g. In this section, we introduce the fundamental concepts and auxiliary results required throughout the paper. Particular attention is devoted to the basic properties of Chebyshev polynomials of the first kind, including their trigonometric representation, recurrence relations, orthogonality structure, and associated weight function. Furthermore, we review several differential properties of these polynomials and discuss the differential operators naturally arising from their orthogonality framework. These preliminaries serve as the foundation for the subsequent derivation of the second-order differential equation and its interpretation within the Sturm–Liouville setting.

The Chebyshev polynomials of the first kind, denoted by $T_n(x)$, are a sequence of orthogonal polynomials defined on the closed interval $[-1, 1]$. They can be uniquely defined by the trigonometric identity

$$T_n(x) = \cos(n \arccos x), \qquad |x| \leq 1.$$

For $n = 0, 1, 2, \ldots$, the first few terms are

$$T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1,$$
$$T_3(x) = 4x^3 - 3x.$$

A fundamental structural property of Chebyshev polynomials of the first kind is that they satisfy the three-term recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1,$$

subject to the initial conditions

$$T_0(x) = 1, \qquad T_1(x) = x.$$

This recurrence relation not only provides an efficient computational mechanism for generating the polynomial sequence but also plays an important role in the investigation of their orthogonality, spectral properties, and associated differential equations.

The Chebyshev polynomials of the first kind form a complete orthogonal system in the weighted Hilbert space

$$L^2\ \ [-1,1], w(x)\,dx\ \ ,$$

where the weight function is given by

$$w(x) = \frac{1}{\sqrt{1-x^2}}.$$

Their orthogonality relation is expressed as

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0, & n \neq m, \\ \pi, & n = m = 0, \\ \dfrac{\pi}{2}, & n = m \neq 0. \end{cases}$$

The function

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

is known as the *Chebyshev weight function.* This weight function plays a fundamental role in the theory of Chebyshev polynomials, as it determines their orthogonality structure and naturally leads to the associated Sturm–Liouville problem. Moreover, it satisfies a Pearson-type differential equation, which establishes a direct connection between the orthogonality properties of the polynomial system and the second-order differential equation satisfied by the Chebyshev polynomials.

The relationship between the Chebyshev polynomials and their derivatives plays a fundamental role in the analysis of their differential properties. One of the most important identities, which follows directly from the trigonometric representation of the polynomials, is given by

$$(1-x^2)T_n'(x) = n\ \ T_{n-1}(x) - xT_n(x)\ \ .$$

This formula establishes a direct connection between the derivative of a Chebyshev polynomial and the polynomial sequence itself.

Consequently, it reveals the intrinsic differential structure underlying the family $\{T_n(x)\}_{n=0}^{\infty}$ and provides a natural starting point for deriving the second-order differential equation satisfied by these polynomials. Moreover, this identity highlights the close relationship between the orthogonality properties of the Chebyshev system and the associated Sturm–Liouville operator, which will be examined in the subsequent sections.

3| Existence of a Second-Order Differential Equation for Cheby-Shev Polynomials of the First Kind

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Using the well-known trigonometric identity $\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta)$, one obtains the three-term recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1.$$

The first few Chebyshev polynomials are

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x.$$

It is well known that the sequence $\{T_n(x)\}_{n=0}^{\infty}$ is orthogonal on the interval $(-1,1)$ with respect to the weight function

$$h(x) = \frac{1}{\sqrt{1-x^2}}, \qquad x \in (-1,1).$$

Furthermore, by introducing the normalization

$$\overline{T}_n(x) = \sqrt{\frac{2}{\pi}}\cos(n \arccos x), \qquad n \geq 1,$$

the resulting system becomes orthonormal in the corresponding weighted Hilbert space.

To derive the differential equation associated with Chebyshev polynomials, we employ the classical Pearson equation. In general, a weight function satisfying

$$\frac{h'(x)}{h(x)} = \frac{p_0 + p_1 x}{q_0 + q_1 x + q_2 x^2}$$

is said to satisfy a Pearson-type relation.

For the Chebyshev weight function

$$h(x) = \frac{1}{\sqrt{1-x^2}},$$

a direct calculation yields

$$h'(x) = \frac{x}{(1-x^2)^{3/2}},$$

and therefore

$$\frac{h'(x)}{h(x)} = \frac{\frac{x}{(1-x^2)^{3/2}}}{\frac{1}{\sqrt{1-x^2}}} = \frac{x}{1-x^2}.$$

Hence, the Chebyshev weight function satisfies the Pearson-type equation

$$\frac{h'(x)}{h(x)} = \frac{x}{1-x^2} = \frac{A(x)}{B(x)}, \tag{1}$$

where

$$A(x) = x, \qquad B(x) = 1 - x^2.$$

This Pearson representation plays a crucial role in establishing the second-order differential equation satisfied by the Chebyshev polynomials and provides the foundation for the subsequent analysis.
Here,

$$A(x) = x, \qquad B(x) = 1 - x^2.$$

Furthermore,

$$\lim_{x\to -1^+} h(x)B(x) = \lim_{x\to -1^+} \frac{1}{\sqrt{1-x^2}}(1-x^2) = \lim_{x\to -1^+} \sqrt{1-x^2} = 0$$

and

$$\lim_{x\to 1} h(x)B(x) = \lim_{x\to 1} \frac{1}{\sqrt{1-x^2}}(1-x^2) = \lim_{x\to 1} \sqrt{1-x^2} = 0.$$

Hence, the boundary conditions required by the Pearson framework are satisfied.

Theorem 1. *Let* $h(x) = \frac{1}{\sqrt{1-x^2}}, \qquad x \in (-1,1)$ *be the Chebyshev weight function. Suppose that* h *satisfies the Pearson equation* $B(x)h'(x) = A(x)h(x)$ *where* $A(x) = x, \qquad B(x) = 1-x^2$*, and* $\lim_{x\to -1^+} B(x)h(x) = 0, \qquad \lim_{x\to 1} B(x)h(x) = 0$*. Then the Chebyshev polynomials of the first kind satisfy the differential equation*

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0.$$

Proof: Since

$$h(x) = \frac{1}{\sqrt{1-x^2}},$$

we obtain

$$h'(x) = \frac{x}{(1-x^2)^{3/2}}.$$

Therefore,

$$(1-x^2)h'(x) = \frac{x}{\sqrt{1-x^2}} = xh(x),$$

which verifies the Pearson equation

$$B(x)h'(x) = A(x)h(x),$$

with

$$A(x) = x, \qquad B(x) = 1-x^2.$$

Moreover,

$$\lim_{x\to -1^+} h(x)B(x) = \lim_{x\to -1^+} \sqrt{1-x^2} = 0,$$

and

$$\lim_{x\to 1} h(x)B(x) = \lim_{x\to 1} \sqrt{1-x^2} = 0.$$

Hence, the boundary conditions required by the Pearson framework are satisfied.

According to the classical theory of orthogonal polynomials, every polynomial sequence associated with a weight function satisfying a Pearson equation is an eigenfunction of a second-order differential operator of Sturm–Liouville type. In particular, the corresponding differential equation is

$$B(x)y'' + \; B'(x) + A(x) \; y' + \lambda_n y = 0.$$

Since

$$B(x) = 1-x^2, \qquad B'(x) = -2x, \qquad A(x) = x,$$

we obtain

$$(1-x^2)y'' - xy' + \lambda_n y = 0.$$

To determine the eigenvalue λ_n, recall that $T_n(x) = 2^{n-1}x^n + \cdots$. Substituting this leading term into the differential equation yields

$$(1-x^2)T_n''(x) - xT_n'(x) = -n^2\, 2^{n-1}x^n + \cdots.$$

Comparing the coefficients of the highest-degree term on both sides gives $\lambda_n = n^2$. Consequently,

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0.$$

Therefore, the Chebyshev polynomials of the first kind satisfy the second-order differential equation

$$(1-x^2)y'' - xy' + n^2y = 0.$$

This completes the proof.

4| Conclusion

In this study, the existence and structure of second-order differential equations for first-type Chebyshev polynomials are investigated within the framework of orthogonality theory and Pearson equations. First, the fundamental properties, trigonometric representations, and orthogonality relations of Chebyshev polynomials are considered, and then it is shown that the weight function corresponding to these polynomials satisfies a Pearson-type differential equation. Using the obtained Pearson relation, the differential operator associated with Chebyshev polynomials is systematically examined, and it is shown that these polynomials are solutions to the second-order linear differential equation of the form:

$$(1-x^2)y''-xy'+n^2y=0$$

Thus, it has been confirmed once again that Chebyshev polynomials are not only an orthogonal polynomial system but also form an important eigenfunction family within the scope of Sturm-Liouville theory.The approach used in this study contributes to a clearer understanding of the structural relationship between weight functions, Pearson equations, and orthogonal polynomials. Furthermore, the results highlight the interaction between differential equation theory and orthogonality theory by revealing the connections between the fundamental concepts of classical orthogonal polynomial theory.

In conclusion, this study provides a holistic perspective on the analytic structure of Chebyshev polynomials and demonstrates how the transition from orthogonality properties to differential equation structure can be achieved through Pearson equations. Future research could explore similar methods for other classical orthogonal polynomial families, as well as investigating the spectral properties of the resulting differential operators and their behavior in various function spaces.

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