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A Class of Exponentially Fitted Finite Difference Scheme for Singularly Perturbed Equations Involving Small Delays in Reaction and Convection Terms

Amar Kumar[1], **Bipin Prasad**[1], **Rakesh Ranjan**[2,*]

[1] Department of Mathematics, Munger University, Munger, Bihar- 811201, India; amar@jrsc.ac.in; vipin.rastogi7@gmail.com.
[2] Government Polytechnic Lakhisarai, Science, Technology and Technical Education Department, Govt. Bihar- 811311, India; 90; ranjan@gmail.com.

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Abstract

A new exponentially fitted finite difference method is introduced for the numerical treatment of a second-order singularly perturbed differential equation (SPDDE) involving small delays in the first derivative and undifferentiated terms. The solution of such equations exhibits left-layer or right-layer behavior in the underlying domain. Taylor's series expansion procedure is used for constructing an equivalent valid version of the original problem and deriving a new three-term finite difference scheme, respectively. The non-uniformity in the solution behavior is resolved by introducing an appropriate exponential fitting parameter in the derived new scheme. The resulting system of equations is solved by the well-known 'discrete invariant algorithm.' Convergence analysis of the fitted method is discussed, and the theory is illustrated by performing numerical experiments on test example problems. Tabulated computational results show the applicability and accuracy of the method. Theory and computation show that the method is able to approximate the solution very well with a second-order convergence rate. The graphical representation of the solution graph for the tested problems illustrates the impact of varied delay shifts on the layer behavior of the solution.

Keywords: Differential-difference equation, Singular perturbation problem, Boundary layer, Stability and convergence, finite difference method.

Corresponding Author: ranjan@gmail.com

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1 | Introduction

A differential-difference equation or delay differential equation (DDE) is an equation in which the system's evolution at a specific time is contingent upon its state at a prior period. This differs from ordinary differential equations (ODEs), in which the derivatives rely solely on the present value of the independent variable. A delay differential equation is classified as singularly perturbed if the highest derivative is multiplied by a small parameter. Delay differential equations emerge in the mathematical modeling of diverse practical phenomena, such as microscale heat transfer[1], hydrodynamics of liquid helium[2], second-sound theory[3], thermoelasticity[4], diffusion in polymers[5], reaction-diffusion equations[6], stability[7], and various models for physiological processes or diseases[8, 9].

A delay differential equation (DDE) is classified as a retarded delay differential equation (RDDE) if the delay argument is absent from the highest order derivative term; otherwise, it is referred to as a neutral delay differential equation (NDDE). By confining it to a category where the largest derivative term is multiplied by a small parameter, we obtain singularly perturbed delay differential equations. This study focuses on second-order singularly perturbed retarded differential delay equations (RDDEs). DDEs are often transformed into ODEs with coefficients dependent on the delay by second-order accurate Taylor series expansions of the delay-involved terms. Asymptotic and numerical methods for delay differential equations (DDEs) are sometimes more complex to implement than those for ordinary differential equations (ODEs) due to the necessity of employing suitable approximations for the retarded arguments $y(x-\delta)$ or $y'(x-\delta)$ and their derivatives, as well as the requirement for the algorithm to address the discontinuities caused by the delay. For more details, one may suggest to go through some of the highly recognized monographs and articles: [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and [24, 25, 9, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48] etc.

The literature reveals a limited number of studies addressing the treatment of a category of SPDDEs with small delays (δ) in both convection and reaction terms. The authors propose a three-term exponentially fitted upwind technique [49]. Rate of convergences are quadratic-accelerated using Richard-extrapolation. The researcher in [50] employed a cubic spline fitted method for small delay singularly perturbed differential equations. The Taylor series expansion has been employed to approximate the delay-containing terms, transforming the main problem into a singularly perturbed problem in ordinary differential equations. This is further processed to generate a tridiagonal scheme and solved by the Thomas algorithm. The theoretical study indicates that the scheme attains a second-order rate of convergence for larger values of the perturbation parameter (ε) and a first-order rate of convergence for smaller values. Results are expressed as maximum absolute point-wise errors. A Shishkin mesh [51] strategy has been proposed by authors of [52] for B-spline collocation techniques in SPDDEs with small delays, which are applied over the underlying domain of the problem. The proposed scheme is of almost second-order rate of convergence. A non-standard finite difference approach is described in [53] to address problems in both the left and right end boundary layers. Practical investigates demonstrate that, for fixed values of the grid points and perturbation parameter, as the delay term increases, the width of the boundary layer decreases. In the work [53], the authors devised a mid-point finite difference approach on a uniform mesh. The authors have examined the stability of the suggested numerical scheme utilizing the maximum principle and barrier functions for solution constraints. The results convergence is occurring uniformly at a linear rate of convergence. The researchers of the study [54] have developed three distinct numerical methods to address a category of SPDDEs that incorporate delayed arguments in convection and reaction terms.

Objectives: Inspired by the literature, our primary objective is to investigate a category of boundary value problems in SPDDEs with small delays in both the convection and reaction components using a straightforward and efficient exponentially fitted finite difference method. The scheme is straightforward and readily customizable by the computer.

This paper introduces an exponentially fitted, finite difference, three-term approach to address a category of Singularly Perturbed Differential-Difference Delay Equations (SPDDEs) with small delays in both the convection and response terms on a uniform mesh. The Taylor series expansion is initially utilized to approximate terms with delay arguments, followed by the introduction of a fitting factor, or artificial viscosity, to the fitted finite difference-based three-term scheme, resulting in the formation of a tridiagonal system. This system has been resolved with the renowned Thomas algorithm. The innovative aspect of the proposed strategy lies in the lack of prior research that addresses this type of problem using fitting factors on a uniform grid. This proposed method can yield highly precise results when the condition $\varepsilon > N$ is met. The results produced by the proposed method have been analyzed in terms of maximum absolute point-wise errors (MAPEs) and rates of convergence (ROCs).

The remaining portion of the paper is arranged as follows: Considered continuous problem is stated in Section:2. The methods of solution is described in the sections 3. Convergence analysis of the fitted method are analyzed in the Section: 4. Considered test example problems with the computational results are presented in the Section: 5. Conclusions are presented in the Section: 6. Paper ends with the bibliography.

2 | Considered Continuous Problem

We consider the following class of 2nd order singularly perturbed DDEs with a small delay in first derivative and undifferentiated terms:

Statement of the problem. To introduce the proposed exponentially fitted finite difference scheme, we examine second-order linear singularly perturbed differential equations with delay in the reaction and convection terms, which can be expressed as:

$$\varepsilon u''(w) + \nu(w)\, u'(w-\delta) + r(w)\, u(w-\delta) + \omega(w)\, u(w) = f(w) \tag{1}$$

$\forall\, w \in (0,1) = \Omega$ under the interval and the conditions:

$$u(w) = \eta(w) \ \text{ on } -\delta \le w \le 0, \text{and} u(w) = \zeta(w) \text{ on} 1 \le w \le 1+\eta, \tag{2}$$

where δ is the small positive delay (negative shift) parameters to be order of $0 < \delta = o(\varepsilon) << 1$, $\varepsilon(0 < \varepsilon << 1)$ is small singular perturbation parameter. The functions $\nu(w), r(w), \omega(w), f(w), \eta(w)$and $\zeta(w)$ are sufficiently smooth in $\forall\, w \in (0,1) = \Omega$. The unique solution of equation (1) with (2) exhibits the layer behavior at the left end (neighbourhood of $u = 0$) of the interval if $\nu(w) - \delta r(w) > 0$ and at the right end (neighbourhood of $u = 1$) of the interval if $\nu(w) - \delta r(w) < 0$. If the shift parameter δ is smaller than epsilon, it is acceptable to use Taylor's series expansion for the term that includes the shift argument. Hence, to obtain an accurate estimation of the term incorporating the delay parameter, the Taylor series expansion is utilized in the following manner:

$$\begin{array}{l} w'(u-\delta) \approx w'(u) - \delta w''(u) + O(\delta^2) \text{ and} \\ w(u-\delta) \approx w(u) - \delta w'(u) + \frac{\delta^2}{2} w''(u) + O(\delta^3) \end{array} \tag{3}$$

Substituting equation (3) into equation (1), we get an asymptotically equivalent singularly perturbed two point boundary value problem(BVP) of the following form:

$$\mu u''(w) + \alpha(w)\, u'(w) + \beta(w)\, u(w) = f(w),\ \forall w \in [0,1] \tag{4}$$

with the BCs

$$u(0) = \eta(0) = \eta_0,\ u(1) = \zeta(1) = \zeta_1, \tag{5}$$

with $0 < \mu = \varepsilon - \delta\nu(u) + \frac{\delta^2}{2} r(u) << 1$, $\alpha(w) = \nu(w) - \delta r(w)$ and $\beta(w) = r(w) + \omega(w)$ where η_0 and ζ_1 are finite constants. Equation (1) is transitionable to equation (4) if δ is sufficiently small, improving computational efficiency. Further details on the validity of this transition can be found in Elsgolt's and Norkin [55] and the articles [56, 57, 38]. Thus, the solution of equation (4) will provide a good approximation to the solution of equation (1).

For the differential equation given in (4), let us represent the differential operator L as $Lu(w) = \mu u''(w) + \alpha(w)u'(w) + \beta(w)u(w)$ [58].

Lemma 1. *(Maximum Principle) Let $\varrho(w)$ be a differentiable function such that $\varrho(0)$ is greater than or equal to 0 and $\varrho(1)$ is greater than or equal to 0. Given $L_\tau \varrho(w) \le 0$ for every w in the interval $(0,1)$, it follows that $\varrho(w)$ is greater than or equal to 0 for all w in the closed interval $[0,1]$.*

Proof: See on [57] □

Lemma 2. *(Stability Principle) Let $u(w)$ denote the solution to the problem described in equation (4). By combining it with equation (5), we obtain*

$$\|u(w)\| \le \chi^{-1}\, \|f\| + \max(|\eta_0|, |\zeta|). \tag{6}$$

where $\|\cdot\|$ is the L_∞ norm given by $\|u(w)\| = \max\limits_{0 \le w \le 1} |u(w)|$ and χ^{-1} is lower of $\alpha(w)$ on $[0,1]$.

Proof: Assume that κ^+ and κ^- are distinct barrier functions, and that $\kappa^{\pm}(w) = \chi^{-1}\|f\| + \max(|\eta_0|, |\zeta|) \pm u(w)$. Applying Lemma-1 to the comparison function $\kappa(w) \pm u(w)$ yields the desired estimate without delay. For more information, please refer to [57]. □

Lemma 3. *The problem represented by equations* (4) - (5) *has a solution denoted as* $u(w)$ *that meets*

$$|u(w)| \leq C\left[1 + exp\left(\frac{-\alpha^* w}{\mu}\right)\right]$$

and

$$\left|u^{(m)}(w)\right| \leq C\left[1 + \mu^{-m} exp\left(\frac{-\alpha^* w}{\mu}\right)\right], \forall m \geq 1,$$

where α^* *is lower bound of* $\alpha(w)$.

Proof: See on [59] or [19]. □

3 | Description of the Proposed fitted Method

This section describes the proposed method for the solution of the problem (2.4) with (2.5) which is equivalent/valid verson to the original problem (2.1) with (2.2).

To describe the method we assume that the function $u(w)$ is smooth on $(0, 1)$ and has the value u_i at uniform mesh points $w_i = ih$ for $i = 0, 1, 2, ..., N$, where $h = \frac{1}{N}$ is the interval of differencing(mesh size).

Further, we let the coefficients of the equation (2.4) are evaluated at the midpoint of the each of the interval $[w_{i-1}, w_{i+1}]$ and the solution of the equation (2.4) with (2.5) is given by:

$$u(w) = A_i + B_i e^{\frac{-\alpha_i(w-w_i)}{\mu}} - \frac{(w - w_i)}{\alpha_i}\beta_i u_i,\ w_{i-1} < w < w_{i+1} \tag{7}$$

where A_i, B_i are arbitrary constants whose values are to be determined.

Using the notations $u(w_{i-1}) = u_{i-1}$, $u(w_i) = u_i$, $u(w_{i+1}) = u_{i+1}$, we have the following from the equation (3.1):

$$u_{i-1} = A_i + B_i e^{\frac{\alpha_i h}{\mu}} + \frac{h}{\alpha_i}\beta_i u_i,\ u_{i+1} = A_i + B_i e^{\frac{-\alpha_i h}{\mu}} - \frac{h}{\alpha_i}\beta_i u_i,\ u_i = A_i + B_i$$

Obviously, $u_{i-1} - 2u_i + u_{i+1} = B_i\left(e^{\frac{\alpha_i h}{\mu}} - 2 + e^{\frac{-\alpha_i h}{\mu}}\right)$.
On solving the above relations for A_i and B_i, we get:

$$B_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\left(e^{\frac{\alpha_i h}{\mu}} - 2 + e^{\frac{-\alpha_i h}{\mu}}\right)},\ A_i = \frac{\left(e^{\frac{\alpha_i h}{\mu}} + \frac{e^{-\alpha_i h}}{\mu}\right)u_i - (u_{i+1} + u_{i-1})}{\left(e^{\frac{\alpha_i h}{\mu}} - 2 + e^{\frac{-\alpha_i h}{\mu}}\right)}$$

Substituting A_i and B_i in equation (3.1), we get:

$$u(y) = \frac{\left(e^{\frac{\alpha_i h}{\mu}} + e^{\frac{-\alpha_i h}{\mu}} - 2e^{-\frac{\alpha_i(w-w_i)}{\mu}}\right)u_i}{\left(e^{\frac{\alpha_i h}{\mu}} - 2 + e^{\frac{-\alpha_i h}{\mu}}\right)} + \frac{\left(e^{-\frac{\alpha_i(w-w_i)}{\mu}} - 1\right)(u_{i+1} + u_{i-1})}{\left(e^{\frac{\alpha_i h}{\mu}} - 2 + e^{\frac{-\alpha_i h}{\mu}}\right)} - \frac{w - w_i}{\alpha_i}\beta_i u_i \tag{8}$$

From equation (3.2), we have:

$$u(w_{i-1}) = \frac{\left(e^{-\frac{\alpha_i h}{\mu}} - e^{\frac{\alpha_i h}{\mu}}\right)u_i + \left(e^{\frac{\alpha_i h}{\mu}} - 1\right)(u_{i+1} + u_{i-1})}{\left(e^{\frac{\alpha_i h}{\mu}} - 2 + e^{\frac{-\alpha_i h}{\mu}}\right)} + \frac{h}{\alpha_i}\beta_i u_i \tag{9}$$

$$u(w_{i+1}) = \frac{\left(e^{\frac{\alpha_i h}{\mu}} - e^{-\frac{\alpha_i h}{\mu}}\right)u_i + \left(e^{-\frac{\alpha_i h}{\mu}} - 1\right)(u_{i+1} + u_{i-1})}{\left(e^{\frac{\alpha_i h}{\mu}} - 2 + e^{\frac{-\alpha_i h}{\mu}}\right)} - \frac{h}{\alpha_i}\beta_i u_i \tag{10}$$

Let $\rho = \frac{\hbar}{\mu}$, then using (3.3) and (3.4) we obtain:

$$u\left(w_{i-1}\right)-u\left(w_{i+1}\right)=\left(u_{i-1}-2u_i+u_{i+1}\right)\frac{\left(e^{\rho\alpha_i}-e^{-\rho\alpha_i}\right)}{\left(e^{\rho\alpha_i}-2+e^{-\rho\alpha_i}\right)}+\frac{2h}{\alpha_i}\beta_i u_i \tag{11}$$

Now using the Taylor's series expansion procedure($u(w_{i+1}) = u_{i+1}$ and $u(w_{i-1}) = u_{i-1}$), we have:

$$u_{i+1}=u_i+hu'_i+\tfrac{h^2}{2!}u''_i+\tfrac{h^3}{3!}u'''_i+\tfrac{h^4}{4!}u_i^{(4)}+\tfrac{h^5}{5!}u_i^{(5)}+\tfrac{h^6}{6!}u_i^{(6)}+\tfrac{h^7}{7!}u_i^{(7)}+\tfrac{h^8}{8!}u_i^{(8)}+O(h^9) \tag{12}$$

$$u_{i-1}=u_i-hu'_i+\tfrac{h^2}{2!}u''_i-\tfrac{h^3}{3!}u'''_i+\tfrac{h^4}{4!}u_i^{(4)}-\tfrac{h^5}{5!}u_i^{(5)}+\tfrac{h^6}{6!}u_i^{(6)}-\tfrac{h^7}{7!}u_i^{(7)}+\tfrac{h^8}{8!}u_i^{(8)}-O(h^9) \tag{13}$$

$$u''_{i+1}=u''_i+hu_i^{(3)}+\tfrac{h^2}{2!}u_i^{(4)}+\tfrac{h^3}{3!}u_i^{(5)}+\tfrac{h^4}{4!}u_i^{(6)}+\tfrac{h^5}{5!}u_i^{(7)}+\tfrac{h^6}{6!}u_i^{(8)}+\tfrac{h^7}{7!}u_i^{(9)}+\tfrac{h^8}{8!}u_i^{(10)}+O(h^{11}) \tag{14}$$

$$u''_{i-1}=u''_i-hu_i^{(3)}+\tfrac{h^2}{2!}u_i^{(4)}-\tfrac{h^3}{3!}u_i^{(5)}+\tfrac{h^4}{4!}u_i^{(6)}-\tfrac{h^5}{5!}u_i^{(7)}+\tfrac{h^6}{6!}u_i^{(8)}-\tfrac{h^7}{7!}u_i^{(9)}+\tfrac{h^8}{8!}u_i^{(10)}-O(h^{11}) \tag{15}$$

Using the above expansion equations (3.6) to (3.9), we have

$$u_{i-1}-2u_i+u_{i+1}=\frac{2h^2}{2!}u''_i+\frac{2h^4}{4!}u_i^{(4)}+\frac{2h^6}{6!}u_i^{(6)}+\frac{2h^8}{8!}u_i^{(8)}+O(h^{10}) \tag{16}$$

$$u''_{i-1}-2u''_i+u''_{i+1}=\frac{2h^2}{2!}u_i^{(4)}+\frac{2h^4}{4!}u_i^{(6)}+\frac{2h^6}{6!}u_i^{(8)}+\frac{2h^8}{8!}u_i^{(10)}+O(h^{12}) \tag{17}$$

and

$$u'_i=\frac{u_{i+1}-u_{i-1}}{2h}+O\left(h^2\right) \tag{18}$$

Substituting the expression for $\frac{h^4}{12}u_i^{(6)}$ from the equation (3.11) in the equation (3.10) and simplifying, we get

$$u_{i-1}-2u_i+u_{i+1}=h^2u''_i+\tfrac{h^2}{30}\left(u''_{i-1}-2u''_i+u''_{i+1}-h^2u_i^{(4)}-\tfrac{h^6}{360}u_i^{(8)}\right)+\tfrac{h^4}{12}u_i^{(4)}+\tfrac{2h^8}{8!}u_i^{(8)}+O(h^{10})$$

Or,

$$u_{i-1}-2u_i+u_{i+1}=\frac{h^2}{30}\left(u''_{i-1}+28u''_i+u''_{i+1}\right)+\frac{h^4}{20}u_i^{(4)}+\mathcal{R} \tag{19}$$

where $\mathcal{R}=-\frac{13h^8}{302400}u_i^{(8)}(\xi)$ for $\xi\in\left[w_{i-1},\,w_{i+1}\right]$.
Differentiating equation (2.4) twice, we obtain the following for $u_i^{(4)}$:

$$u_i^{(4)}=\frac{f''_i}{\mu}-\frac{\alpha_i f'_i}{\mu^2}+\left[\frac{\alpha_i^2}{\mu^2}-\frac{(2\alpha'_i+\beta_i)}{\mu}\right]u''_i+\left[\frac{\alpha_i\left(\alpha'_i+\beta_i\right)}{\mu^2}-\frac{(\alpha''_i+2\beta'_i)}{\mu}\right]u'_i+\left[\frac{\alpha_i\beta'_i}{\mu^2}-\frac{\beta''_i}{\mu}\right]u_i \tag{20}$$

From the equation (2.4), we get

$$\mu u''_{i+1}=-\alpha_{i+1}u'_{i+1}-\beta_{i+1}u_{i+1}+f_{i+1} \tag{21}$$

$$\mu u''_i=-\alpha_i u'_i-\beta_i u_i+f_i \tag{22}$$

$$\mu u''_{i-1}=-\alpha_{i-1}u'_{i-1}-\beta_{i-1}u_{i-1}+f_{i-1} \tag{23}$$

The non-symmetric finite difference approximations for first order derivatives u'_{i+1} and u'_{i-1} are given by:

$$u'_{i+1}=\frac{u_{i-1}-4u_i+3u_{i+1}}{2h}-\frac{u_{i-1}-2u_i+u_{i+1}}{h}+O\left(h^2\right) \tag{24}$$

$$u'_{i-1}=\frac{-3u_{i-1}+4u_i-u_{i+1}}{2h}+\frac{u_{i-1}-2u_i+u_{i+1}}{h}+O\left(h^2\right) \tag{25}$$

Substituting the expressions for u''_i, u''_{i+1}, and u''_{i-1} obtained by using the equations (3.12), (3.18) and (3.19) in the equations (3.16), (3.15) and (3.17) respectively, and the expression for $u_i^{(4)}$ from the equation (3.14) in the equation (3.13) and then simplifying, we get

$$\begin{aligned}
&\mu\left(\tfrac{u_{i-1}-2u_i+u_{i+1}}{h^2}\right)+\tfrac{\alpha_{i-1}}{30}\left(\tfrac{-3u_{i-1}+4u_i-u_{i+1}}{2h}+\tfrac{u_{i-1}-2u_i+u_{i+1}}{h}\right)+\tfrac{\beta_{i-1}}{30}u_{i-1}+\tfrac{28\alpha_i}{30}\left(\tfrac{u_{i+1}-u_{i-1}}{2h}\right)+\\
&\tfrac{28\beta_{i-1}}{30}u_i+\tfrac{\beta_{i+1}}{30}u_{i+1}+\tfrac{\alpha_{i+1}}{30}\left(\tfrac{u_{i-1}-4u_i+3u_{i+1}}{2h}-\tfrac{u_{i-1}-2u_i+u_{i+1}}{h}\right)-\tfrac{h^2}{20}\left[\tfrac{\alpha_i\left(\alpha'_i+\beta_i\right)}{\mu}-(\alpha''_i+2\beta'_i)\right]*\\
&\left(\tfrac{u_{i+1}-u_{i-1}}{2h}\right)+\tfrac{h^2\alpha_i}{20}\left[\tfrac{\alpha_i^2}{\mu^2}-\tfrac{\left(2\alpha'_i+\beta_i\right)}{\mu}\right]\left(\tfrac{u_{i+1}-u_{i-1}}{2h}\right)+\left[\tfrac{h^2\beta_i}{20}\left(\tfrac{\alpha_i^2}{\mu^2}-\tfrac{\left(2\alpha'_i+\beta_i\right)}{\mu}\right)-\right.\\
&\left.\tfrac{h^2}{20}\left(\tfrac{\alpha_i\beta'_i}{\mu}-\beta''_i\right)\right]u_i=\tfrac{1}{30}\left[f_{i-1}+28f_i+f_{i+1}\right]+\tfrac{h^2}{20}\left(f''_i-\tfrac{\alpha_i f'_i}{\mu}\right)+\tfrac{h^2 f_i}{20}\left[\tfrac{\alpha_i^2}{\mu^2}-\tfrac{\left(2\alpha'_i+\beta_i\right)}{\mu}\right]
\end{aligned} \tag{26}$$

Now by making use of a constant fitting parameter $\sigma(\rho)$ in the above scheme (3.20), we obtain

$$\begin{aligned}&\sigma(\rho)\,\mu\left(\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}\right)+\frac{\alpha_{i-1}}{30}\left(\frac{-3u_{i-1}+4u_i-u_{i+1}}{2h}+\frac{u_{i-1}-2u_i+u_{i+1}}{h}\right)+\frac{\beta_{i-1}}{30}u_{i-1}+\\&\frac{28\alpha_i}{30}\left(\frac{u_{i+1}-u_{i-1}}{2h}\right)+\frac{28\beta_{i-1}}{30}u_i+\frac{\beta_{i+1}}{30}u_{i+1}+\frac{\alpha_{i+1}}{30}\left(\frac{u_{i-1}-4u_i+3u_{i+1}}{2h}-\frac{u_{i-1}-2u_i+u_{i+1}}{h}\right)-\\&\frac{h^2}{20}\left[\frac{\alpha_i\left(\alpha_i'+\beta_i\right)}{\mu}-\left(\alpha_i''+2\beta_i'\right)\right]\left(\frac{u_{i+1}-u_{i-1}}{2h}\right)+\frac{h^2\alpha_i}{20}\left[\frac{\alpha_i^2}{\mu^2}-\frac{\left(2\alpha_i'+\beta_i\right)}{\mu}\right]\left(\frac{u_{i+1}-u_{i-1}}{2h}\right)+\\&\left[\frac{h^2\beta_i}{20}\left(\frac{\alpha_i^2}{\mu^2}-\frac{\left(2\alpha_i'+\beta_i\right)}{\mu}\right)-\frac{h^2}{20}\left(\frac{\alpha_i\beta_i'}{\mu}-\beta_i''\right)\right]u_i=\frac{1}{30}\left[f_{i-1}+28f_i+f_{i+1}\right]+\frac{h^2}{20}\left(f_i''-\frac{\alpha_i f_i'}{\mu}\right)+\\&\frac{h^2 f_i}{20}\left[\frac{\alpha_i^2}{\mu^2}-\frac{\left(2\alpha_i'+\beta_i\right)}{\mu}\right]\end{aligned}\tag{27}$$

where the values of the parameter $\sigma(\rho)$ is be determined.
Multiplying the above equation(3.21) by h and taking the limit as $h\to 0$, we get

$$\frac{\sigma(\rho)}{\rho}\lim_{h\to 0}\left[u_{i+1}-2u_i+u_{i-1}\right]=\left(\frac{\alpha(0)}{2}+\frac{\rho^2}{40}\left(\alpha(0)\right)^3\right)\lim_{h\to 0}\left[u_{i-1}-u_{i+1}\right]\tag{28}$$

Now, an use of the equation (3.5) in (3.22) gives the required value of the fitting parameter $\sigma(\rho)$ as:

$$\sigma(\rho)=\frac{\rho}{2}\left(\alpha(0)+\frac{\rho^2}{20}\left(\alpha(0)\right)^3\right)\left[\frac{1}{\tanh\left(\frac{\alpha(0)\rho}{2}\right)}\right]\tag{29}$$

Finally, by making use of the equation (3.21) and the value of $\sigma(\rho)$ obtained in the equation (3.23), we get the following fitted three-term finite difference scheme:

$$L_h\equiv E_iu_{i-1}-F_iu_i+G_iu_{i+1}=R_i,\ \ (i=1,2,3,...,N-1)\tag{30}$$

with

$$E_i=\frac{\mu\sigma(\rho)}{h^2}-\frac{\alpha_{i-1}}{60h}+\frac{\beta_{i-1}}{30}-\frac{28\alpha_i}{60h}-\frac{\alpha_{i+1}}{60h}+\frac{h}{40}\left[\frac{\alpha_i\left(\alpha_i'+\beta_i\right)}{\mu}-\left(\alpha_i''+2\beta_i'\right)\right]-\frac{h\alpha_i}{40}\left[\frac{\alpha_i^2}{\mu^2}-\frac{\left(2\alpha_i'+\beta_i\right)}{\mu}\right]$$

$$F_i=\frac{2\mu\sigma(\rho)}{h^2}-\frac{28\beta_i}{30}-\frac{h^2\beta_i}{20}\left[\frac{\alpha_i^2}{\mu^2}-\frac{\left(2\alpha_i'+\beta_i\right)}{\mu}\right]+\frac{h^2}{20}\left[\frac{\alpha_i\beta_i'}{\mu}-\beta_i''\right]$$

$$G_i=\frac{\mu\sigma(\rho)}{h^2}+\frac{\alpha_{i-1}}{60h}+\frac{\beta_{i+1}}{30}+\frac{28\alpha_i}{60h}+\frac{\alpha_{i+1}}{60h}-\frac{h}{40}\left[\frac{\alpha_i\left(\alpha_i'+\beta_i\right)}{\mu}-\left(\alpha_i''+2\beta_i'\right)\right]+\frac{h\alpha_i}{40}\left[\frac{\alpha_i^2}{\mu^2}-\frac{\left(2\alpha_i'+\beta_i\right)}{\mu}\right]$$

$$R_i=\frac{1}{30}\left[f_{i-1}+28f_i+f_{i+1}\right]+\frac{h^2}{20}\left(f_i''-\frac{\alpha_i f_i'}{\mu}\right)+\frac{h^2 f_i}{20}\left[\frac{\alpha_i^2}{\mu^2}-\frac{\left(2\alpha_i'+\beta_i\right)}{\mu}\right]$$

The scheme (3.24) with (2.5) provides a tri-diagonal algebraic system of $(N-1)$ equations with $(N-1)$ unknowns u_1 to u_{N-1}. We will solve this system by 'Discrete Invariant Imbedding algorithm' (also known as "Thomas Algorithm") described in [38, 60, 36]. It is known that the "Thomas algorithm" provides a numerically stable and convergent technique for solving the tridiagonal system of equations of the form(3.24)-(2.5) under the following conditions [61]:

$$F_i>(E_i+G_i),|E_i|\leq|G_i|,E_i>0,G_i>0\tag{31}$$

By considering the functions $\alpha(w)=\alpha$ and $\beta(w)=\beta$ as constant functions on [0,1] where α and β are finite constants, one can easily observe that these conditions (3.25) hold true for the problems whose solution either exhibits left layer or right layer behavior. In case of the problems whose solution exhibits left layer and right layer behaviour, all the conditions (3.25) are satisfied when $\left(-\beta-\frac{h^2\beta}{20}\left[\frac{\alpha^2}{\varepsilon^2}-\frac{\beta}{\varepsilon}\right]\right)>0$ where $\beta(w)=\beta<0$.

4 | Convergence Analysis of the Proposed fitted Method

In this section, we will discuss the convergence/error analysis of the proposed fitted method [62, 63].
Definition 1. Let

$$\tau_i[u]\equiv L_hu(w_i)-L_\tau u(w_i),\ \ i=1,2,...,N-1,$$

where u is a continuously differentiable function on $I = [0,1]$ with L_h as the discrete difference operator. Then the difference problem (3.24) with (2.5) is *consistent* with the difference problem (2.4) with (2.5) if

$$|\tau_i[u]| \to 0 \ \ as \ h \to 0.$$

The quantities $\tau_i[u]$, $i = 1, 2, ..., N-1$, are said to be the local truncation (or local discretization) errors.

Definition 2. The difference problem (3.24) with (2.5) has local $r^{th}-$ order accuracy if, for sufficiently smooth data, a positive constant C exists independent of h and ε such that

$$\max_{1 \le i \le N} |\tau_i[u]| \le Ch^r.$$

The consistency of the difference problem (3.24)-(2.5) with (2.4)-(2.5) and its locally second-order accuracy is shown by the following lemma.

Lemma 4. If $u \in C^4(I)$, then

$$|\tau_i| \le \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \frac{28\alpha h^2}{180} \left| u^{(3)}(w) \right| \right\} + O\left(h^3\right);\ i = 1, 2, ..., N-1.$$

proof. By definition

$$\tau_i = \sigma(\rho)\mu \left\{ \left(\tfrac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \right) - u_i'' \right\} + \tfrac{\alpha_{i-1}}{30} \left\{ \left(\tfrac{-3u_{i-1} + 4u_i - u_{i+1}}{2h} + \tfrac{u_{i-1} - 2u_i + u_{i+1}}{h} \right) - u_{i-1}' \right\} + \tfrac{28\alpha_i}{30} \left\{ \left(\tfrac{u_{i+1} - u_{i-1}}{2h} \right) - u_i' \right\} + \tfrac{\alpha_{i+1}}{30} \left\{ \left(\tfrac{u_{i-1} - 4u_i + 3u_{i+1}}{2h} - \tfrac{u_{i-1} - 2u_i + u_{i+1}}{h} \right) - u_{i+1}' \right\}.$$

$$\Rightarrow \tau_i = \sigma(\rho)\mu \left\{ \tfrac{h^2}{12} u_i^{(4)} + \tfrac{h^4}{360} u_i^{(6)} + ... \right\} + \tfrac{\alpha_{i-1}}{30} \left\{ hu_i'' - \tfrac{h^2}{3} u_i^{(3)} + ... \right\} + \tfrac{28\alpha_i}{30} \left\{ \tfrac{h^2}{6} u_i^{(3)} + \tfrac{h^4}{120} u_i^{(5)} + ... \right\} + \tfrac{\alpha_{i+1}}{30} \left\{ -hu_i'' - \tfrac{h^2}{3} u_i^{(3)} + ... \right\}$$

$$\Rightarrow |\tau_i| \le \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \tfrac{\sigma h^2 \mu}{12} \left| u^{(4)}(w) \right| \right\} + \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \tfrac{\alpha h}{30} \left| u''(w) \right| \right\} + \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \tfrac{28\alpha h^2}{180} \left| u^{(3)}(w) \right| \right\} - \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \tfrac{\alpha h}{30} \left| u''(w) \right| \right\}$$

$$\Rightarrow |\tau_i| \le \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \frac{\sigma h^2 \mu}{12} \left| u^{(4)}(w) \right| \right\} + \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \frac{28\alpha h^2}{180} \left| u^{(3)}(w) \right| \right\}$$

Using the relation (3.23), we get

$$\Rightarrow |\tau_i| \le \left(\frac{h^3 \alpha(0) \coth\left(\frac{h\alpha(0)}{2\mu} \right)}{24} \right) \left(1 + \frac{h^2 \alpha^2(0)}{20\mu^2} \right) \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \left| u^{(4)}(w) \right| \right\} + \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \tfrac{28\alpha h^2}{180} \left| u^{(3)}(w) \right| \right\}$$

Using the expansion of $\coth\left(\frac{h\alpha(0)}{2\mu} \right)$, we get

$$\Rightarrow |\tau_i| \le \left(\frac{h^3}{24} + O\left(h^5\right) \right) \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \left| u^{(4)}(w) \right| \right\} + \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \frac{28\alpha h^2}{180} \left| u^{(3)}(w) \right| \right\}$$

$$\Rightarrow |\tau_i| \le \max_{w_{i-1} \le w \le w_{i+1}} \left\{ \frac{28\alpha h^2}{180} \left| u^{(3)}(w) \right| \right\} + O(h^3)$$

$$\Rightarrow |\tau_i| \le O\left(h^2\right);\ i = 1, 2, ..., N-1. \tag{32}$$

Thus, the desired result is obtained.

Definition 3. (Uniform Convergence) Let $u(w)$ be the exact solution and u^N be the approximate solution obtained by any numerical method/scheme for the problem under consideration. If there exist a relation of the form $\|u - u^N\| \le Ch^r$, where $C > 0$ and $r > 0$ are independent of μ and of the mesh h, then we say that the method/scheme is of $r - th$ order accurate and converges uniformly to $u(w)$ with respect to the norm $\|.\|$ [64].

Theorem 1. Let $u(w)$ be the analytical solution of the equation (2.4) with (2.5) [and hence (2.1) with (2.2)] and let $u^N(w)$ be the approximate solution obtained by the discretized scheme (3.24) with (2.5). Then for sufficiently small h, the error component satisfies the uniform error estimate: $\|u - u^N\| \le C^* h^2$ with C^* a positive constant.

Proof. Clearly the system of equations (3.24) has the following matrix-vector form [63]:

$$Vu = D \tag{33}$$

where $V = (a_{ij})\,,\ 1 \le i,j \le N-1$ is a tri-diagonal matrix of order $(N-1)$, with

$$a_{i,i+1} = \frac{\mu\sigma(\rho)}{h^2} + \frac{\alpha_{i-1}}{60h} + \frac{\beta_{i+1}}{30} + \frac{28\alpha_i}{60h} + \frac{\alpha_{i+1}}{60h} - \frac{h}{40}\left[\frac{\alpha_i\left(\alpha_i'+\beta_i\right)}{\mu} - (\alpha_i'' + 2\beta_i')\right] + \frac{h\alpha_i}{40}\left[\frac{\alpha_i^2}{\mu^2} - \frac{\left(2\alpha_i'+\beta_i\right)}{\mu}\right]$$

$$a_{i,i} = -\left(\frac{2\mu\sigma(\rho)}{h^2} - \frac{28\beta_i}{30} - \frac{h^2\beta_i}{20}\left[\frac{\alpha_i^2}{\mu^2} - \frac{\left(2\alpha_i'+\beta_i\right)}{\mu}\right] + \frac{h^2}{20}\left[\frac{\alpha_i\beta_i'}{\mu} - \beta_i''\right]\right)$$

$$a_{i,i-1} = \frac{\mu\sigma(\rho)}{h^2} - \frac{\alpha_{i-1}}{60h} + \frac{\beta_{i-1}}{30} - \frac{28\alpha_i}{60h} - \frac{\alpha_{i+1}}{60h} + \frac{h}{40}\left[\frac{\alpha_i\left(\alpha_i'+\beta_i\right)}{\mu} - (\alpha_i'' + 2\beta_i')\right] - \frac{h\alpha_i}{40}\left[\frac{\alpha_i^2}{\mu^2} - \frac{\left(2\alpha_i'+\beta_i\right)}{\mu}\right]$$

and $D = (d_i)$ is a column vector with $d_i = \frac{1}{30}\left[f_{i-1} + 28f_i + f_{i+1}\right] + \frac{h^2}{20}\left(f_i'' - \frac{\alpha_i f_i'}{\mu}\right) + \frac{h^2 f_i}{20}\left[\frac{\alpha_i^2}{\mu^2} - \frac{\left(2\alpha_i'+\beta_i\right)}{\mu}\right]$, for $i = 1, 2, ..., (N-1)$, with the local truncation error τ_i given by

$$|\tau_i| \le O\left(h^2\right) \tag{34}$$

from the equation (4.1).
We also have

$$V\overline{u} - \tau(h) = D \tag{35}$$

where $\overline{u} = \left(\overline{u}_0, \overline{u}_1, \overline{u}_2, ..., \overline{u}_N\right)^w$ and $\tau(h) = \left(\tau_1(h), \tau_2(h), ..., \tau_N(h)\right)^y$ denote the actual solution and the local truncation error respectively.
From equations (4.2) and (4.4), we have

$$V\left(\overline{u} - u\right) = \tau(h) \tag{36}$$

Thus, the error equation is

$$VE = \tau(h) \tag{37}$$

where $E = \overline{u} - u = \left(e_0, e_1, e_2, ..., e_N\right)^t$.
Let S_i^* be the sum of elements of the i^{th} row of V, then we have

$$S_1^* = \sum_{j=1}^{N-1} a_{1,j} = \frac{-\mu\sigma(\rho)}{h^2} + \frac{(28\beta_1+\beta_2)}{30} + \frac{(\alpha_0+28\alpha_1+\alpha_2)}{60h} + \left[\frac{\alpha_1^2}{\mu^2} - \frac{\left(2\alpha_1'+\beta_1\right)}{\mu}\right] * \left[\frac{h^2\beta_1}{20} + \frac{h\alpha_1}{40}\right] - \frac{h^2}{20}\left[\frac{\alpha_1\beta_1'}{\mu} - \beta_1''\right] - \frac{h}{40}\left[\frac{\alpha_1\left(\alpha_1'+\beta_1\right)}{\mu} - (\alpha_1'' + 2\beta_1')\right]$$

$$S_{N-1}^* = \sum_{j=1}^{N-1} a_{N-1,j} = \frac{-\mu\sigma(\rho)}{h^2} + \frac{(28\beta_{N-1}+\beta_{N-2})}{30} - \frac{(\alpha_{N-2}+28\alpha_{N-1}+\alpha_N)}{60h} + \frac{h^2}{20}\left[\frac{\alpha_{N-1}\beta_{N-1}'}{\mu} - \beta_{N-1}''\right] - \left[\frac{\alpha_{N-1}^2}{\mu^2} - \frac{\left(2\alpha_{N-1}'+\beta_{N-1}\right)}{\mu}\right] * \left[\frac{h^2\beta_{N-1}}{20} + \frac{h\alpha_{N-1}}{40}\right] - \frac{h}{40}\left[\frac{\alpha_{N-1}\left(\alpha_{N-1}'+\beta_{N-1}\right)}{\mu} - \left(\alpha_{N-1}'' + 2\beta_{N-1}'\right)\right]$$

$$S_i^* = \sum_{j=1}^{N-1} a_{i,j} = \frac{1}{30}\left[\beta_{i-1} + 28\beta_i + \beta_{i+1}\right] + O(h^2)$$
$$= b_i = B_{i_0};\ i = 2\,(1)\,N-2;$$
$$\text{where } B_{i_0} = b_i = \frac{1}{30}\left[\beta_{i-1} + 28\beta_i + \beta_{i+1}\right].$$

Since $0 < \epsilon << 1$, the matrix V is irreducible and monotone for sufficiently small h. Thus V^{-1} exists with the non-negative elements. Hence, from equation(4.6) we obtain

$$E = V^{-1}\tau(h) \tag{38}$$

$$||E|| \le ||V^{-1}||||\tau(h)|| \tag{39}$$

Let $\overline{a}_{ki}$ be the $(ki)^{th}$ elements of V^{-1}. Since $\overline{a}_{ki} \ge 0$, from the operations of matrices we have.

$$\sum_{i=1}^{N-1} \overline{a}_{ki} S_i^* = 1;\ k = 1, 2, ..., N-1 \tag{40}$$

Therefore, it follows that

$$\sum_{i=1}^{N-1} \overline{a}_{ki} \leq \frac{1}{\underset{0 \leq i \leq N-1}{min} S_i^*} = \frac{1}{B_{i_0}} \leq \frac{1}{|B_{i_0}|} \tag{41}$$

for some i_0 between 1 and $(N-1)$ and $B_{i_0} = b_i$.
Therefore, from equations(4.3),(4.7) and (4.9) we obtain

$$e_j = \sum_{i=1}^{N-1} \overline{a}_{ki} \tau_i(h);\ j = 1(1)N-1$$

which implies

$$e_j \leq \frac{O\left(h^2\right)}{|b_i|};\ j = 1(1)N-1 \tag{42}$$

By making use of the definitions and the equation(4.11) we obtain

$$||E|| = o(h^2)$$

Thus the method proposed is of second order convergence.

5 | Test Example Problems and the Computational Results

Here, we will demonstrate the performance of the present method by solving four test example problems; presenting the results in terms of maximum absolute pointwise errors(MAPEs) and the numerical order of convergence(NOC). We will also present the graphical solutions with varying values of the delay parameter to examine the effect of small delay on the left layer and right layer behavior of the solutions. Since the exact solution of the considered test example problems are not available, the well known double mesh principle [14] is used to calculate the MAPEs and NOC. The formula for finding MAPEs (E_ε^N) is given by:

$$E_\varepsilon^N = \max_{0 \leq i \leq N} \left| u_i^N - u_{2i}^{2N} \right|,\ i = 0, 1, 2, ..., N,$$

where u_i^N and u_{2i}^{2N} are the numerical solutions obtained on the mesh $w_i = w_0 + ih$, $i = 0, 1, 2, ..., N$ and on the mesh obtained by halving of the original mesh size h with $2N$ number of mesh intervals respectively. Furthermore, the NOC (R_ε^N) is obtained using the formula: $R_\varepsilon^N = \log_2 \left| \frac{E_\varepsilon^N}{E_\varepsilon^{2N}} \right|$.

Example problems exhibiting left layer behaviour. In this subsection we will solve two SPDDEs whose solutions exhibite the left layer behaviour in $[0, 1]$.
Example 1. Firstly, we consider the following problem

$$\begin{aligned}&\varepsilon u''(w) + (1+w)u'(w-\delta) + \exp(-2w)u(w-\delta) - 2\exp(-w)u(w) = 0;\\ &\text{with}\, u(0) = 1\, \text{and}\, u(1) = 0\end{aligned}$$

Example 2. We consider the following problem

$$\begin{aligned}&\varepsilon w''(u) + (1+u)w'(u-\delta) + \sin(2u)w(u-\delta) - \exp(-u)w(u) =\\ &\sin(2u) + 3\exp(-u);\, \text{with}\, w(0) = 1\, \text{and}\, w(1) = 1\end{aligned}$$

We solved these two example problems for various values of δ, ε and N, and tabulated the computational results in the Tables 1 to 2. Table 1 and 2 presents the results in terms of MAPEs for different N and ε with $\delta = 0.5 * \varepsilon$. Further, a simple look on the numerical order of convergence(NOC) (R_ε^N) presented in the Table 1 for example-1 and the Table 2 for example-2 clearly reveals that the method proposed is of the second order convergence rate. In order to examine the effect of small delay (δ) on the left layer behaviour of the solutions for the example-1 and 2 we plotted the graphs for $\varepsilon = 0.1, 0.01$ in the figures 1 to 2 and 4 to 5 respectively with fixed $N = 100$ and the varying values of δ where $\delta = o(\varepsilon)$. By the simple observation of the figures 1 to 2 and 4 to 5 we easily confirm that there is no significant effect of small delay δ on the left layer behavior of the solution when $\delta = o(\varepsilon)$ with ε small. That is, when the values of ε tends to zero the effect of small delay δ on the left layer behavior becoming negligible for all $\delta = o(\varepsilon)$. Thus the effect of small delay δ can be ignored in case when $\delta = o(\varepsilon)$ with ε sufficiently small. For the example-1 qnd 2, the effect of the delay δ is to increase and decrease the width of the left boundary layers. Plotted the Loglog graphs for the MAPEs (E_ε^N) for different N with various ε in the Figures 3 and 6. From these Figures 3 and 6, one can easily visualize the higher accuracy of the suggested method for the increasing ε more clearly.

Table 1. MAPEs E_ε^N and NOC R_ε^N for different ε and N with varying $\delta = 0.5 * \varepsilon$ for example-1

$\varepsilon \downarrow$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
2^{-1}	3.1988E-04 2.0025	7.9830E-05 2.0000	1.9958E-05 2.0003	4.9884E-06 1.9998	1.2473E-06 2.0008	3.1165E-07
2^{-2}	8.0466E-04 2.0091	1.9990E-04 2.0022	4.9899E-05 2.0007	1.2469E-05 2.0006	3.1160E-06 1.9992	7.7944E-07
2^{-3}	1.8181E-03 2.0123	4.5066E-04 2.0085	1.1200E-04 2.0015	2.7970E-05 2.0002	6.9914E-06 1.9995	1.7484E-06
2^{-4}	4.1914E-03 2.0952	9.8093E-04 2.0193	2.4197E-04 2.0078	6.0167E-05 2.0005	1.5037E-05 2.0007	3.7574E-06
2^{-5}	1.1206E-02 2.3647	2.1757E-03 2.0856	5.1259E-04 2.0231	1.2611E-04 2.0058	3.1400E-05 2.0009	7.8450E-06
2^{-6}	2.3277E-02 2.0204	5.7377E-03 2.3691	1.1106E-03 2.0807	2.6254E-04 2.0253	6.4495E-05 2.0047	1.6071E-05
2^{-7}	3.5431E-02 1.5744	1.1897E-02 2.0336	2.9058E-03 2.3718	5.6140E-04 2.0781	1.3295E-04 2.0265	3.2633E-05
2^{-8}	4.1268E-02 1.1831	1.8175E-02 1.5944	6.0190E-03 2.0410	1.4626E-03 2.3733	2.8229E-04 2.0770	6.6905E-05
2^{-9}	4.3164E-02 1.0252	2.1209E-02 1.2026	9.2149E-03 1.6056	3.0280E-03 2.0449	7.3378E-04 2.3740	1.4155E-04

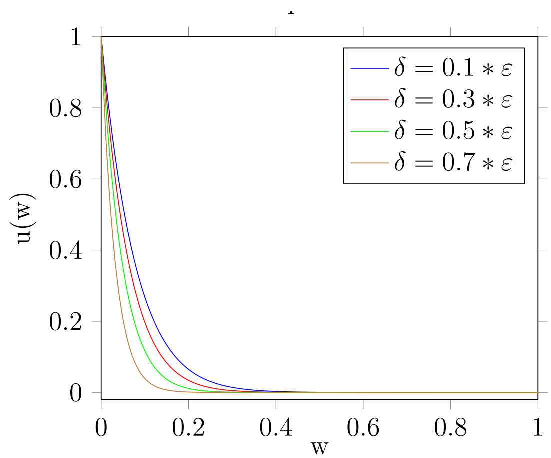

Figure 1. Graphical solutions for different δ with $N = 100$ and $\varepsilon = 0.1$ for the example 1

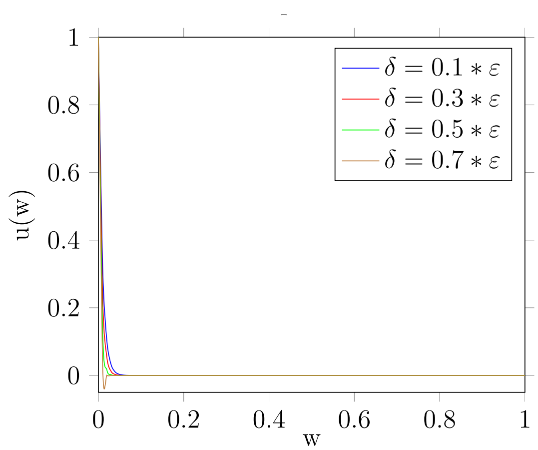

Figure 2. Graphical solutions for different δ with $N = 100$ and $\varepsilon = 0.01$ for the example 1

Example problems exhibiting right layer behaviour. We performed the numerical experiments on the following two SPDDEs type whose solutions exhibite right layer behaviour in the interval $[0, 1]$:
Example 3. We consider the following problem

$$-\varepsilon u''(w) + (1+w)u'(w-\delta) - \exp(-2w)u(w-\delta) + \exp(-w)u(w) = 0;$$
$$\text{with } u(0) = 1 \text{ and } u(1) = -1$$

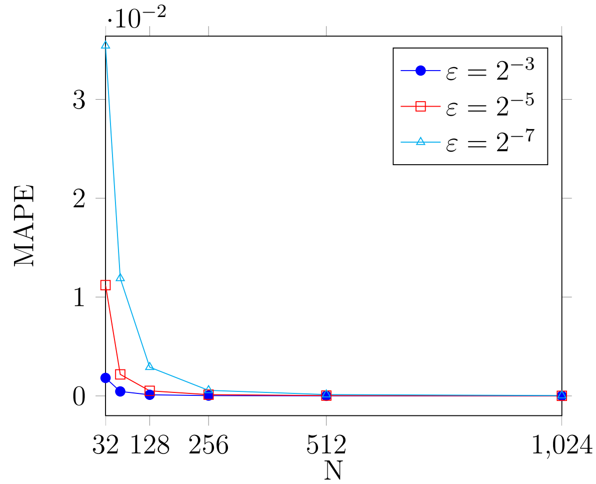

FIGURE 3. Loglog plot of the MAPE E_ε^N of example 1 for different value of N

TABLE 2. MAPEs E_ε^N and NOC R_ε^N for different ε and N with varying $\delta = 0.5 * \varepsilon$ for example-2

$\varepsilon \downarrow$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
2^{-1}	7.5565E-04 2.0019	1.8866E-04 2.0001	4.7161E-05 2.0009	1.1783E-05 2.0000	2.9458E-06 1.9597	7.5729E-07
2^{-2}	1.8099E-03 2.0008	4.5223E-04 2.0012	1.1296E-04 2.0003	2.8234E-05 1.9993	7.0621E-06 1.9952	1.7714E-06
2^{-3}	3.9676E-03 2.0176	9.7989E-04 2.0010	2.4480E-04 2.0012	6.1149E-05 2.0001	1.5286E-05 2.0095	3.7965E-06
2^{-4}	8.5063E-03 2.0084	2.1142E-03 2.0171	5.2234E-04 2.0043	1.3020E-04 1.9998	3.2555E-05 1.9985	8.1472E-06
2^{-5}	2.1625E-02 2.2999	4.3915E-03 1.9933	1.1030E-03 2.0168	2.7255E-04 2.0042	6.7940E-05 2.0007	1.6977E-05
2^{-6}	4.2144E-02 1.9248	1.1100E-02 2.3073	2.2427E-03 1.9872	5.6566E-04 2.0167	1.3979E-04 2.0042	3.4847E-05
2^{-7}	6.1910E-02 1.5183	2.1613E-02 1.9399	5.6332E-03 2.3045	1.1403E-03 1.9910	2.8685E-04 2.0168	7.0884E-05
2^{-8}	7.1786E-02 1.1746	3.1802E-02 1.5370	1.0959E-02 1.9486	2.8392E-03 2.3032	5.7525E-04 1.9932	1.4449E-04
2^{-9}	7.5313E-02 1.0319	3.6834E-02 1.1902	1.6142E-02 1.5480	5.5204E-03 1.9533	1.4255E-03 2.3026	2.8894E-04

Example 4. Finally, consider the following problem

$$-\varepsilon u''(w) + (1+w)u'(w-\delta) - \exp(-2w)u(w-\delta) + \exp(-w)u(w) = \exp(x-1);$$
$$\text{with}\, u(0) = 1 \,\text{and}\, u(1) = -1$$

We solved these two example problems for various values of δ, ε and N, and tabulated the computational results in the Tables 3 to 4. Table 3 and 4 presents the results in terms of MAPEs for different N and ε with $\delta = 0.5 * \varepsilon$. Further, a simple look on the numerical order of convergence(NOC) (R_ε^N) presented in the Table 3 for example-3 and the Table 4 for example-4 clearly reveals that the method proposed is of the second order convergence rate. In order to examine the effect of small delay (δ) on the right layer behaviour of the solutions for the example-3 and 4 we plotted the graphs for $\varepsilon = 0.1, 0.01$ in the figures 7 to 8 and 10 to 11 respectively with fixed $N = 100$ and the varying values of δ where $\delta = o(\varepsilon)$. By the simple observation of the figures 7 to 8 and 10 to 11 we

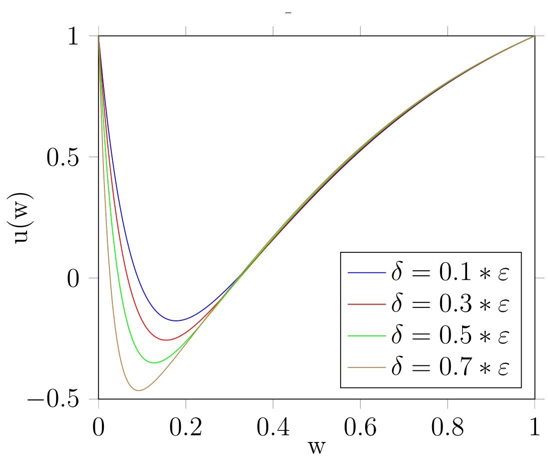

FIGURE 4. Graphical solutions of example 2 for different value of δ with $N = 100$, $\varepsilon = 0.1$

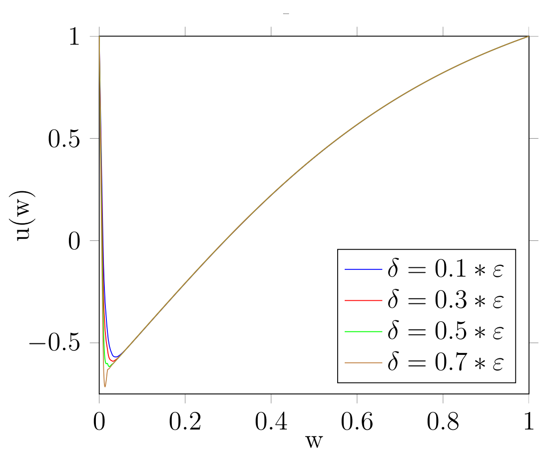

FIGURE 5. Graphical solutions of example 2 for different value of δ with $N = 100$, $\varepsilon = 0.01$

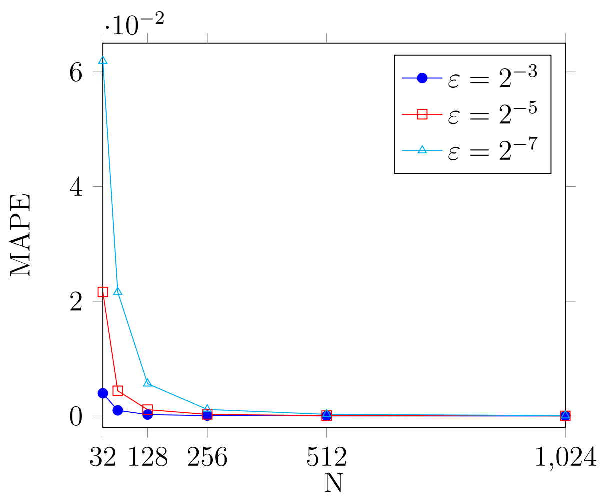

FIGURE 6. Loglog plot of the MAPE E_ε^N of example 2 for different value of N

easily confirm that there is no significant effect of small delay δ on the right layer behavior of the solution when $\delta = o(\varepsilon)$ with ε small. That is, when the values of ε tends to zero the effect of small delay δ on the right layer behavior becoming negligible for all $\delta = o(\varepsilon)$. Thus the effect of small delay δ can be ignored in case when $\delta = o(\varepsilon)$ with ε sufficiently small. For the example-3 and 4, the effect of the delay δ is to decrease and increase the width of the right boundary layers. Plotted the Loglog graphs for the MAPEs (E_ε^N) for different N with different ε in the Figures 9 and 12. From this Figures 9 and 12, one can easily visualize the higher accuracy of the sugested method for the increasing ε more clearly.

6 | Conclusion

We developed a new method for finding the solution of the considered SPDDEs exhibiting left-layer or right-layer behavior and implemented it on four standard widely discussed problems. To show the applicability, accuracy, and efficiency of the method, we tabulated the results in terms of MAPEs (maximum absolute pointwise errors) and the NOC (numerical order of convergence). Both the theory and the numerical computation state that

TABLE 3. MAPEs E_ε^N and NOC R_ε^N for different ε and N with varying $\delta = 0.5 * \varepsilon$ for example-3

$\varepsilon \downarrow$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
2^{-1}	5.7093E-05 2.0005	1.4268E-05 1.9994	3.5686E-06 2.0018	8.9104E-07 2.0025	2.2237E-07 1.9989	5.5634E-08
2^{-2}	3.4139E-04 2.0022	8.5216E-05 2.0001	2.1303E-05 1.9998	5.3266E-06 2.0006	1.3311E-06 2.0045	3.3173E-07
2^{-3}	9.6897E-04 2.0119	2.4026E-04 2.0029	5.9943E-05 2.0005	1.4981E-05 2.0001	3.7450E-06 2.0000	9.3625E-07
2^{-4}	1.9943E-03 2.0155	4.9326E-04 2.0123	1.2227E-04 2.0032	3.0499E-05 2.0007	7.6210E-06 1.9997	1.9056E-06
2^{-5}	4.7054E-03 2.2018	1.0228E-03 2.0191	2.5233E-04 2.0125	6.2539E-05 2.0027	1.5606E-05 2.0007	3.8997E-06
2^{-6}	1.2778E-02 2.4061	2.4108E-03 2.2020	5.2395E-04 2.0196	1.2922E-04 2.0126	3.2025E-05 2.0029	7.9902E-06
2^{-7}	2.6490E-02 2.0243	6.5118E-03 2.4095	1.2257E-03 2.2022	2.6636E-04 2.0195	6.5697E-05 2.0122	1.6286E-05
2^{-8}	3.8506E-02 1.5131	1.3491E-02 2.0333	3.2958E-03 2.4127	6.1898E-04 2.2026	1.3447E-04 2.0191	3.3175E-05
2^{-9}	4.3530E-02 1.1460	1.9670E-02 1.5260	6.8304E-03 2.0414	1.6593E-03 2.4148	3.1117E-04 2.2028	6.7592E-05

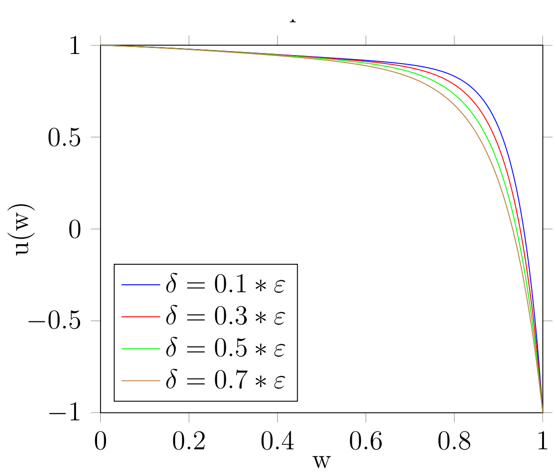

FIGURE 7. Graphical solutions of example 3 for different value of δ with $N = 100$, $\varepsilon = 0.1$

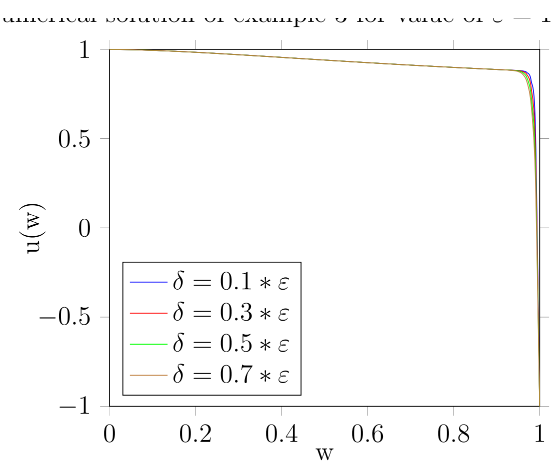

FIGURE 8. Graphical solutions of example 3 for different value of δ with $N = 100$, $\varepsilon = 0.01$

the present method is capable of approximating the solution very well by producing second-order convergent solutions. Furthermore, we plotted the solution graphs for various values of δ, while keeping N and ε fixed, to examine how the delay δ affects the behavior of the solution in both the left and right layers. One can easily observe from figures 1 to 2 and 4 to 5, for example, problems 1 and 2, respectively, that there is no significant effect of small delay δ on the left layer behavior of the solution when $\delta = o(\varepsilon)$ with ε sufficiently small. As the values of ε decrease, the effect of a small delay δ on the behavior of the left layer of the solution becomes negligible. Also, the effect of the delay δ is to increase and decrease the width of the left boundary layers. Plotted the Loglog graphs for the MAPEs (E_ε^N) for different N with various ε in the Figures 3 and 6. From these Figures 3 and 6, one can easily visualize the higher accuracy of the suggested method for the increasing ε more clearly. In case of problems with the right layer behavior of the solutions, an examination of figures 7 to 8

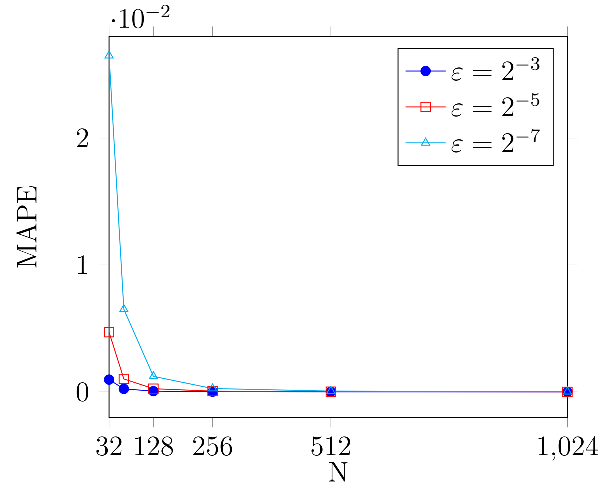

FIGURE 9. Loglog plot of the MAPE E_ε^N of example 3 for different value of N

TABLE 4. MAPEs E_ε^N and NOC R_ε^N for different ε and N with varying $\delta = 0.5 * \varepsilon$ for example-4

$\varepsilon \downarrow$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
2^{-1}	5.4255E-05 1.9997	1.3567E-05 1.9994	3.3931E-06 2.0033	8.4634E-07 1.9978	2.1191E-07 1.9954	5.3148E-08
2^{-2}	3.0077E-04 2.0019	7.5096E-05 2.0001	1.8773E-05 1.9996	4.6947E-06 1.9998	1.1738E-06 2.0248	2.8845E-07
2^{-3}	8.5048E-04 2.0102	2.1112E-04 2.0014	5.2729E-05 2.0008	1.3175E-05 1.9996	3.2947E-06 2.0008	8.2324E-07
2^{-4}	1.7722E-03 2.0164	4.3803E-04 2.0105	1.0871E-04 2.0028	2.7124E-05 2.0003	6.7795E-06 2.0018	1.6928E-06
2^{-5}	4.0862E-03 2.1665	9.1019E-04 2.0143	2.2530E-04 2.0108	5.5905E-05 2.0021	1.3956E-05 2.0013	3.4858E-06
2^{-6}	1.0977E-02 2.3868	2.0988E-03 2.1659	4.6769E-04 2.0150	1.1571E-04 2.0108	2.8712E-05 2.0027	7.1648E-06
2^{-7}	2.2430E-02 1.9989	5.6118E-03 2.3923	1.0689E-03 2.1648	2.3838E-04 2.0161	5.8933E-05 2.0106	1.4625E-05
2^{-8}	3.2197E-02 1.4889	1.1471E-02 2.0111	2.8458E-03 2.3969	5.4035E-04 2.1641	1.2056E-04 2.0169	2.9789E-05
2^{-9}	3.6081E-02 1.1258	1.6534E-02 1.5057	5.8227E-03 2.0213	1.4343E-03 2.3997	2.7180E-04 2.1639	6.0653E-05

and 10 to 11 for example problems 3 and 4, respectively, shows there is no significant effect of small delay δ on the right layer behavior of the solution when $\delta = o(\varepsilon)$ with ε sufficiently small. As the values of ε decrease, the effect of small delay δ on the right layer behavior of the solution becomes negligible. Furthermore, the effect of the delay δ is to decrease and increase the width of the right boundary layers. Plotted the Loglog graphs for the MAPEs (E_ε^N) for different N with different ε in the Figures 9 and 12. From this Figures 9 and 12, one can easily visualize the higher accuracy of the sugested method for the increasing ε more clearly. Finally, it is concluded that the current method seems to be one of the best options for approximating the solution of the considered class of SPDDEs.

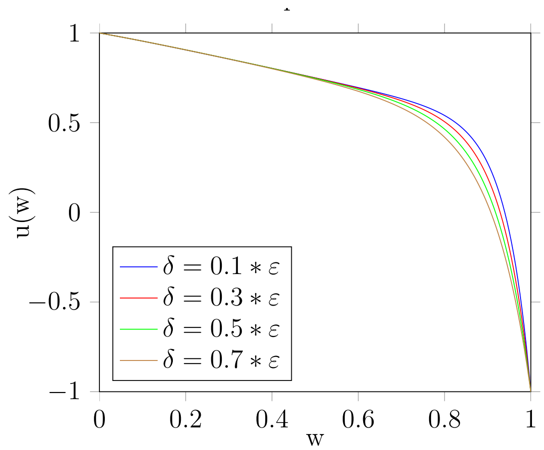

Figure 10. Graphical solutions of example 4 for different value of δ with $N = 100$, $\varepsilon = 0.1$

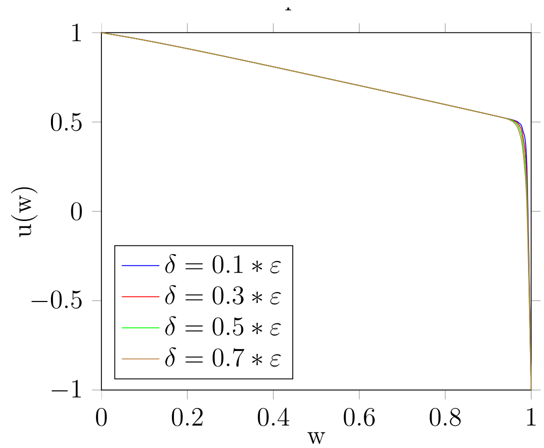

Figure 11. Graphical solutions of example 4 for different value of δ with $N = 100$, $\varepsilon = 0.01$

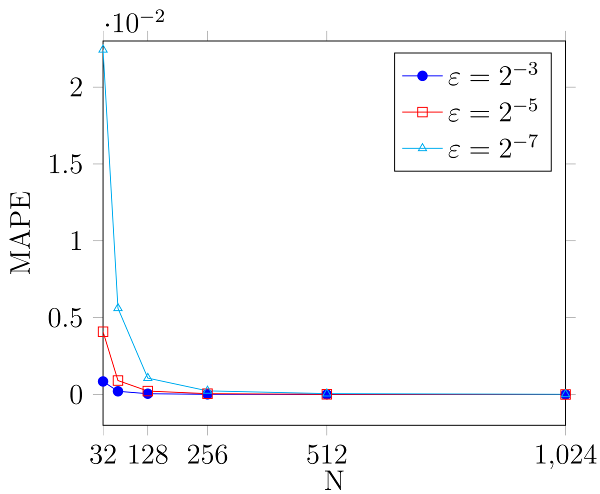

Figure 12. Loglog plot of the MAPE E_{ε}^{N} of example 4 for different value of N

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Author Contribution

A. Kumar: conceptualization and editing. B. Prasad: methodology and software. R. Ranjan: writing and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

References

[1] Tzou, D. Y. (1997). *Macro- to microscale heat transfer: The lagging behavior*. Journal of the National Cancer Institute. https://catalog.library.vanderbilt.edu/discovery/fulldisplay/alma991028993169703276/01VAN_INST:vanui

[2] Joseph, D. D., & Preziosi, L. (1989). Heat waves. *Reviews of modern physics, 61*(1), 41-73. https://doi.org/10.1103/RevModPhys.61.41

[3] Joseph, D. D., & Preziosi, L. (1990). Addendum to the paper "Heat waves". *Reviews of modern physics, 62*(2), 375–391. https://dept.aem.umn.edu/~faculty/joseph/archive/docs/HeatWavesAdd.pdf

[4] Ezzat, M. A., Othman, M. I., & El-Karamany, A. M. (2002). State space approach to two-dimensional generalized thermo-viscoelasticity with two relaxation times. *International journal of engineering science, 40*(11), 1251-1274. https://doi.org/10.1016/S0020-7225(02)00012-5

[5] Liu, Q., Wang, X., & De Kee, D. (2005). Mass transport through swelling membranes. *International journal of engineering science, 43* (19-20), 1464-1470. https://doi.org/10.1016/j.ijengsci.2005.05.010

[6] Bestehorn, M., & Grigorieva, E. V. (2004). Formation and propagation of localized states in extended systems. *Annalen der physik, 516*(7-8), 423-431. https://doi.org/10.1002/andp.200451607-806

[7] Burton, T. A. (2005). Fixed points, stability, and exact linearization. *Nonlinear analysis: Theory, methods & applications, 61*(5), 857-870. https://doi.org/10.1016/j.na.2005.01.079

[8] Sirisha, L., & Reddy, Y. N. (2014). Fitted second order scheme for singularly perturbed differential-difference equations. *American journal of numerical analysis, 2*(5), 136–143. https://doi.org/10.12691/ajna-2-5-1

[9] Mackey, M. C., & Glass, L. (1977). Oscillation and chaos in physiological control systems. *Science, 197*(4300), 287-289. https://doi.org/10.1126/science.267326

[10] Balachandran, B., Kalmár-Nagy, T., & Gilsinn, D. E. (2009). *Delay differential equations*. Berlin: Springer. https://doi.org/10.1007/978-0-387-85595-0

[11] Cooke, K. L. (1963). Differential—difference equations. In *International symposium on nonlinear differential equations and nonlinear mechanics* (pp. 155-171). Academic Press. https://doi.org/10.1016/B978-0-12-395651-4.50022-2

[12] Bellman, R. E., & Cooke, K. L. (1963). *Differential-difference equations*. Academic Press, New York. https://www.rand.org/content/dam/rand/pubs/reports/2006/R374.pdf

[13] Bellen, A., & Zennaro, M. (2003). *Numerical methods for delay differential equations*. OUP Oxford. https://doi.org/10.1093/acprof:oso/9780198506546.001.0001

[14] Doolan, E. P., Miller, J. J., & Schilders, W. H. (1980). *Uniform numerical methods for problems with initial and boundary layers*. Boole Press. https://research.tue.nl/en/publications/uniform-numerical-methods-for-problems-with-initial-and-boundary-/

[15] Wille, D. R. (1989). *The numerical solution of delay-differential equations*. The University of Manchester (United Kingdom). https://search.proquest.com/openview/6dcf691994bf221842d7c8087d749b10/1?pq-origsite=gscholar&cbl=2026366&diss=y

[16] Driver, R. D. (2012). *Ordinary and delay differential equations*. Springer Science & Business Media. https://doi.org/10.1007/978-1-4684-9467-9

[17] Eckhaus, W. (2011). *Matched asymptotic expansions and singular perturbations*. Elsevier. https://shop.elsevier.com/books/matched-asymptotic-expansions-and-singular-perturbations/eckhaus/978-0-444-10438-0

[18] Kuang, Y. (1993). *Delay differential equations: with applications in population dynamics*. Academic Press, INC. https://api.pageplace.de/preview/DT0400.9780080960029_A23536877/preview-9780080960029_A23536877.pdf

[19] Miller, J. J., O'riordan, E., & Shishkin, G. I. (1996). *Fitted numerical methods for singular perturbation problems: Error estimates in the maximum norm for linear problems in one and two dimensions*. World Scientific. https://www.gbv.de/dms/goettin-gen/211493112.pdf

[20] Nayfeh, A. H. (1973). *Perturbation methods*. John Wiley & Sons. https://books.google.com/books?id=eh6RmWZ51NIC&printsec=frontcover#v=onepage&q&f=false

[21] O'Malley, R. E. (1974). *Introduction to singular perturbations*. Academic Press. https://cir.nii.ac.jp/crid/1970867909825496889

[22] Kevorkian, J., & Cole, J. D. (2013). *Perturbation methods in applied mathematics*. Springer Science & Business Media. https://doi.org/10.1007/978-1-4757-4213-8

[23] Bender, C. M., & Orszag, S. A. (1978). *Advanced mathematical methods for scientists and engineers*. McGraw-Hill. https://books.google.com/books?id=-yQXwhE6iWMC

[24] Derstine, M. W., Gibbs, H. M., Hopf, F. A., & Kaplan, D. L. (1982). Bifurcation gap in a hybrid optically Bistable system. *Physical review A, 26*(6), 3720. https://doi.org/10.1103/PhysRevA.26.3720

[25] Longtin, A., & Milton, J. G. (1988). Complex oscillations in the human pupil light reflex with "mixed" and delayed feedback. *Mathematical biosciences, 90*(1-2), 183-199. https://doi.org/10.1016/0025-5564(88)90064-8

[26] Kadalbajoo, M. K., & Reddy, Y. (1989). Asymptotic and numerical analysis of singular perturbation problems: A survey. *Applied mathematics and computation, 30*(3), 223-259. https://doi.org/10.1016/0096-3003(89)90054-4

[27] Lange, C. G., & Miura, R. M. (1994). Singular perturbation analysis of boundary-value problems for differential-difference equations. V. Small shifts with layer behavior. *SIAM journal on applied mathematics, 54*(1), 249-272. https://www.jstor.org/stable/2102177

[28] Lange, C. G., & Miura, R. M. (1994). Singular perturbation analysis of boundary-value problems for differential-difference equations. VI. Small shifts with rapid oscillations. *SIAM journal on applied mathematics, 54*(1), 273-283. https://doi.org/10.1137/S0036139992228119

[29] Lange, C. G., & Miura, R. M. (1982). Singular perturbation analysis of boundary value problems for differential-difference equations. *SIAM journal on applied mathematics, 42*(3), 502-531. https://doi.org/10.1137/0142036

[30] Lange, C. G., & Miura, R. M. (1985). Singular perturbation analysis of boundary value problems for differential-difference equations III. Turning point problems. *SIAM journal on applied mathematics, 45*(5), 708-734. https://doi.org/10.1137/0145042

[31] Stein, R. B.(1967). Some models of neuronal variability. *Biophysical journal, 7*(1), 37-68. https://doi.org/10.1016/S0006-3495(67)86574-3

[32] Villasana, M., & Radunskaya, A. (2003). A delay differential equation model for tumor growth. *Journal of mathematical biology, 47*(3), 270-294. https://doi.org/10.1007/s00285-003-0211-0

[33] Kaur, M., & Arora, G. (2017). A review on singular perturbed delay differential equations. *International journal of current advanced research, 6*(3), 2341-2346. https://doi.org/10.24327/ijcar.2017.2346.0005

[34] McCartin, B. J. (2001). Exponential fitting of the delayed recruitment/renewal equation. *Journal of computational and applied mathematics, 136*(1-2), 343-356. https://doi.org/10.1016/S0377-0427(00)00625-7

[35] Kadalbajoo, M. K., & Sharma, K. K. (2006). An ε-uniform convergent method for a general boundary-value problem singularly perturbed differential-difference equations: Small shifts of mixed type with layer behavior. *Journal of computational* for *methods in science and engineering, 6*(1-4), 39-55. https://doi.org/10.3233/JCM-2006-61-405

[36] Kadalbajoo, M. K., & Sharma, K. K. (2002). Numerical analysis of boundary-value problems for singularly-perturbed differential-difference equations with small shifts of mixed type. *Journal of optimization theory and applications, 115*(1), 145-163. https://doi.org/10.1023/A:1019681130824

[37] Kadalbajoo, M. K., & Kumar, D. (2008). Fitted mesh B-spline collocation method for singularly perturbed differential–difference equations with small delay. *Applied mathematics and computation, 204*(1), 90-98. https://doi.org/10.1016/j.amc.2008.05.140

[38] Kadalbajoo, M. K., & Sharma, K. K. (2003). An ϵ-uniform fitted operator method for solving boundary-value problems for singularly perturbed delay differential equations: Layer behavior. *International journal of computer mathematics, 80*(10), 1261-1276. https://doi.org/10.1080/0020716031000103394

[39] Kadalbajoo, M. K., & Sharma, K. K. (2005). Numerical treatment of a mathematical model arising from a model of neuronal variability. *Journal of mathematical analysis and applications, 307*(2), 606-627. https://doi.org/10.1016/j.jmaa.2005.02.014

[40] Kadalbajoo, M. K., & Kumar, D. (2010). A computational method for singularly perturbed nonlinear differential-difference equations with small shift. *Applied mathematical modelling, 34*(9), 2584-2596. https://doi.org/10.1016/j.apm.2009.11.021

[41] Kumar, V., & Sharma, K. K. (2008). An optimized B-spline method for solving singularly perturbed differential difference equations with delay as well as advance. *Neural, parallel and scientific computations, 16*(3), 371. https://dynamicpublishers.com/Neural/NPSC2008/28-aggarwal_sharma_accepted_NPSC.pdf

[42] Kadalbajoo, M. K., & Patidar, K. C. (2002). A survey of numerical techniques for solving singularly perturbed ordinary differential equations. *Applied mathematics and computation, 130*(2-3), 457-510. https://doi.org/10.1016/S0096-3003(01)00112-6

[43] Kumar, M., Singh, P., & Mishra, H. K. (2007). A recent survey on computational techniques for solving singularly perturbed boundary value problems. *International journal of computer mathematics, 84*(10), 1439-1463. https://doi.org/10.1080/00207160701295712

[44] Sharma, K. K., Rai, P., & Patidar, K. C. (2013). A review on singularly perturbed differential equations with turning points and interior layers. *Applied mathematics and computation, 219*(22), 10575-10609. https://doi.org/10.1016/j.amc.2013.04.049

[45] Kadalbajoo, M. K., & Gupta, V. (2010). A brief survey on numerical methods for solving singularly perturbed problems. *Applied mathematics and computation, 217*(8), 3641-3716. https://doi.org/10.1016/j.amc.2010.09.059

[46] Malge, S., & Lodhi, R. K. (2022). Survey on numerical and analytical methods for singularly perturbed differential-difference equations. *Mathematics in engineering, science & aerospace (MESA), 13*(2), 355-364. https://nonlinearstudies.com/index.php/mesa/article/view/2596

[47] Kaur, J., & Kaur, B. (2023). A review on singularly perturbed problems. *International journal of research and analytical reviews, 10*(3), 310–320. https://www.ijrar.org/papers/IJRAR23C2671.pdf

[48] Duressa, G. F., & Debela, H. G. (2021). Numerical solution of singularly perturbed differential difference equations with mixed parameters. *Journal of mathematical modeling (JMM), 9*(4), 691-705. https://doi.org/10.22124/jmm.2021.18365.1576

[49] Woldaregay, M. M., & Duressa, G. F. (2022). Fitted numerical scheme for singularly perturbed differential equations having two small delays. *Caspian journal of mathematical sciences, 11*(1), 98-114. https://doi.org/10.22080/cjms.2020.19425.1547

[50] Ranjan, R. (2023). A novel fitted spline method for the numerical treatment of singularly perturbed differential equations having small delays. *Journal of applied mathematics and computing, 69*(6), 4645-4664. https://doi.org/10.1007/s12190-023-01939-8

[51] Shishkin, G. I. (1988). A difference scheme for a singularly perturbed equation of parabolic type with discontinuous boundary conditions. *USSR computational mathematics and mathematical physics, 28*(6), 32-41. https://doi.org/10.1016/0041-5553(88)90039-0

[52] Kumar, D., & Kadalbajoo, M. (2012). Numerical treatment of singularly perturbed delay differential equations using B-spline collocation method on Shishkin mesh. *Journal of numerical analysis, industrial and applied mathematics, 7*(3-4), 73-90. https://jnaiam.org/wp-content/uploads/2024/02/Numerical-Treatment-of-Singularly-Perturbed-Delay-Differential-Equations-Using-B-Spline-Collocation-Method-on-Shishkin-Mesh.pdf

[53] Woldaregay, M. M., & Duressa, G. F. (2021). Robust numerical scheme for solving singularly perturbed differential equations involving small delays. *Applied mathematics e-notes, 21*, 622-633. https://www.math.nthu.edu.tw/~amen/2021/AMEN-200826.pdf

[54] Woldaregay, M. M. (2022). Solving singularly perturbed delay differential equations via fitted mesh and exact difference method. *Research in mathematics, 9*(1), 2109301. https://doi.org/10.1080/27684830.2022.2109301

[55] Norkin, S. B. (1973). *Introduction to the theory and application of differential equations with deviating arguments.* Academic Press. https://books.google.com/books?id=l4XU6QKpx2sC&printsec=frontcover#v=onepage&q&f=false

[56] Kadalbajoo, M. K., & Sharma, K. K. (2004). Numerical analysis of singularly perturbed delay differential equations with layer behavior. *Applied mathematics and computation, 157*(1), 11-28. https://doi.org/10.1016/j.amc.2003.06.012

[57] Rao, R. N., & Chakravarthy, P. P. (2014). An exponentially fitted tridiagonal finite difference method for singularly perturbed differential-difference equations with small shift. *Ain shams engineering journal, 5*(4), 1351-1360. https://doi.org/10.1016/j.asej.2014.07.001

[58] Protter, M. H., & Weinberger, H. F. (2012). Maximum principles in differential equations. Springer. https://www.gbv.de/dms/ilmenau/toc/302301399.PDF

[59] Kellogg, R. B., & Tsan, A. (1978). Analysis of some difference approximations for a singular perturbation problem without turning points. *Mathematics of computation, 32*(144), 1025-1039. https://doi.org/10.1090/S0025-5718-1978-0483484-9

[60] Kadalbajoo, M. K., & Reddy, Y. N. (1986). Numerical integration of a class of singular perturbation problems. *Journal of optimization theory and applications, 51*(3), 441-452. https://doi.org/10.1007/BF00940284

[61] Nichols, N. K. (1989). On the numerical integration of a class of singular perturbation problems. *Journal of optimization theory and applications, 60*(3), 439-452. https://doi.org/10.1007/BF00940347

[62] Kiltu, G. G., Duressa, G. F., & Bullo, T. A. (2020). Numerical treatment of singularly perturbed delay reaction-diffusion equations. *International journal of engineering, science and technology, 12*(1), 15-24. http://dx.doi.org/10.4314/ijest.v12i1.2

[63] Aziz, T., & Khan, A. (2002). A spline method for second-order singularly perturbed boundary-value problems. *Journal of computational and applied mathematics, 147*(2), 445-452. https://doi.org/10.1016/S0377-0427(02)00479-X

[64] O'Riordan, E., & Stynes, M. (1991). A globally uniformly convergent finite element method for a singularly perturbed elliptic problem in two dimensions. *Mathematics of computation, 57*(195), 47-62. https://doi.org/10.2307/2938662